Quantum dichotomies and coherent thermodynamics beyond first-order asymptotics

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TEAM-NET

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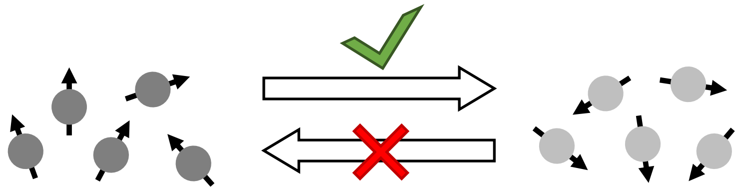
Overview

Quantum dichotomies

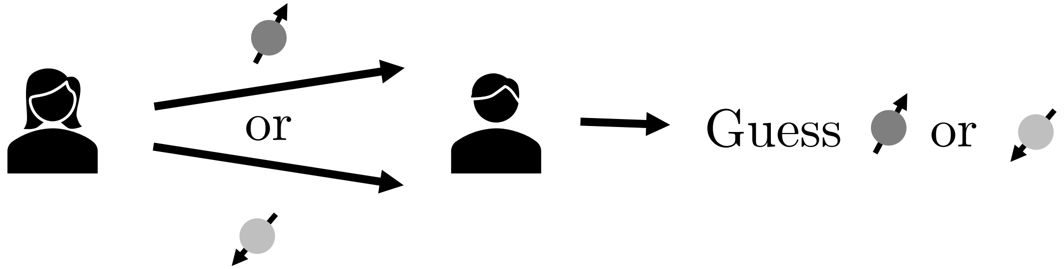
Quantum thermodynamics | **Quantum information theory**

Problem: thermodynamic transformations between non-equilibrium states



- Coherent: superpositions of energy eigenstates
- Asymptotic: number of systems $n \to \infty$

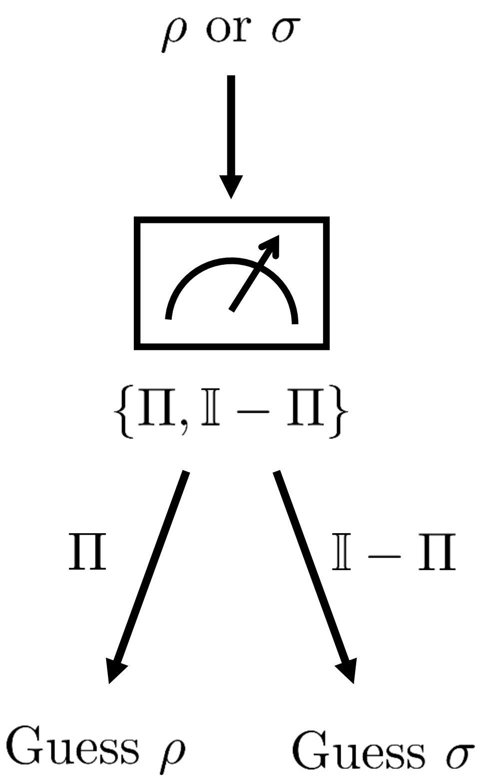
Problem: Quantum hypothesis testing



Quantum dichotomies and coherent thermodynamics beyond first-order asymptotics

Background

Quantum hypothesis testing



		It was	
		ρ	σ
You guessed	ρ	Success $p_{\mathrm{I}} = \mathrm{Tr}(\Pi\rho)$	Type II error $\epsilon_{\mathrm{II}} = \mathrm{Tr}(\Pi\sigma)$
	σ	Type I error $\epsilon_{\mathrm{I}} = 1 - \mathrm{Tr}(\Pi\rho)$	Success $p_{\mathrm{II}} = 1 - \mathrm{Tr}(\Pi\sigma)$

Optimal type-II error given type-I error

$$\beta_x(\rho\|\sigma) := \min_{\Pi}\{\epsilon_{\mathrm{II}} \mid \epsilon_{\mathrm{I}} = x\}$$

Quantum dichotomies

Quantum dichotomy: (ρ, σ)

Blackwell order: $(\rho_1, \sigma_1) \succ (\rho_2, \sigma_2) \iff \exists \mathcal{E} : \; \mathcal{E}(\rho_1) = \rho_2 \;\text{ and }\; \mathcal{E}(\sigma_1) = \sigma_2$

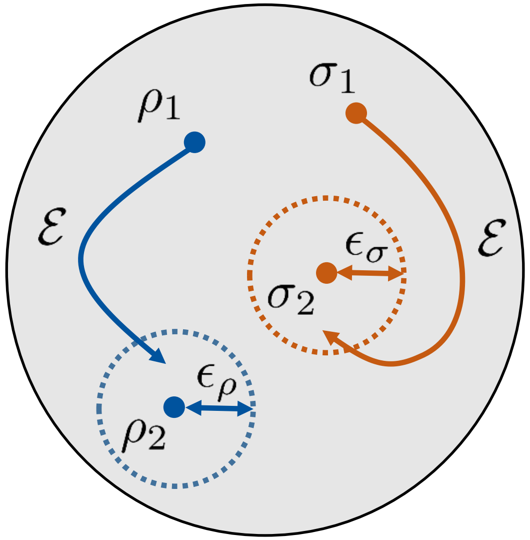
Approximate order: $(\rho_1, \sigma_1) \succ_{(\epsilon_\rho, \epsilon_\sigma)} (\rho_2, \sigma_2) \iff \exists \mathcal{E} : \; \delta(\mathcal{E}(\rho_1), \rho_2) \leq \epsilon_\rho \;\text{ and }\; \delta(\mathcal{E}(\sigma_1), \sigma_2) \leq \epsilon_\sigma$

Relation between dichotomies and hypothesis testing

$[\rho_1, \sigma_1] = [\rho_2, \sigma_2] = 0$ *	$[\rho_1, \sigma_1] \neq 0, \quad [\rho_2, \sigma_2] = 0$
$\forall_{x \in (\epsilon_\rho, 1)} \; \beta_x(\rho_1 \Vert \sigma_1) \leq \beta_{x-\epsilon_\rho}(\rho_2 \Vert \sigma_2) + \epsilon_\sigma$ $\Updownarrow$ $(\rho_1, \sigma_1) \succ_{(\epsilon_\rho, \epsilon_\sigma)} (\rho_2, \sigma_2)$	$\forall_{x \in (\epsilon_\rho, 1)} \; \beta_x(\rho_1 \Vert \sigma_1) \leq \beta_{x-\epsilon_\rho}(\rho_2 \Vert \sigma_2) + \epsilon_\sigma$ $\Uparrow$ $(\rho_1, \sigma_1) \succ_{(\epsilon_\rho, \epsilon_\sigma)} (\rho_2, \sigma_2)$ $\Uparrow$ $\forall_{x \in (\epsilon_\rho, 1)} \; \widetilde{\beta}_x(\rho_1 \Vert \sigma_1) \leq \widetilde{\beta}_{x-\epsilon_\rho}(\rho_2 \Vert \sigma_2) + \epsilon_\sigma$

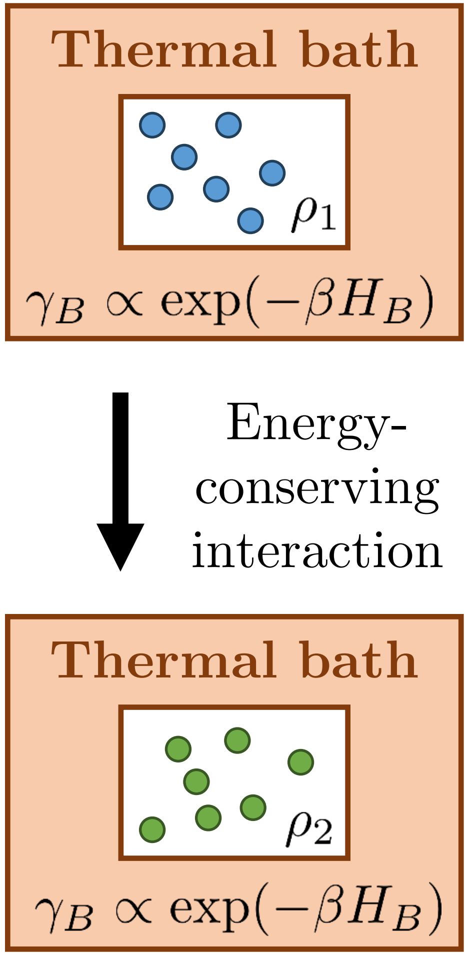
$\widetilde{\beta}_x(\rho \Vert \sigma) = \beta_x(\mathcal{D}_\sigma(\rho) \Vert \sigma)$

$\mathcal{D}_\sigma$: pinching w.r.t. σ



*J. Math. Phys. **57**, 122202 (2016)

Quantum thermodynamics



Thermal operations framework*

$$\rho_1 \xrightarrow[\text{TO}]{} \rho_2 \quad \Longleftrightarrow \quad \rho_2 = \mathrm{Tr}_{B'}\left(U\left(\rho_1 \otimes \gamma_B\right)U^\dagger\right)$$

$$\text{with } [U, H \otimes \mathbb{I}_B + \mathbb{I} \otimes H_B] = 0$$

Approximate thermal operations**

$$\rho_1 \xrightarrow[\text{TO}]{\epsilon} \rho_2 \quad \Longleftrightarrow \quad \rho_1 \xrightarrow[\text{TO}]{} \widetilde{\rho}_2 \text{ and } \delta(\widetilde{\rho}_2, \rho_2) \leq \epsilon$$

Thermal state of the system: $\gamma \propto \exp(-\beta H)$

*Nat. Commun. **4**, 2059 (2013)
Quantum **2, 108 (2018)

Quantum thermodynamics

Relation between thermodynamic transformations, dichotomies, and hypothesis testing

Incoherent states: $[\rho_1,\gamma]=[\rho_2,\gamma]=0$ *	Final incoherent state: $[\rho_2,\gamma]=0$
$\rho_1 \xrightarrow[\text{TO}]{\epsilon} \rho_2$	$\underset{x\in(\epsilon,1)}{\forall}\ \beta_x(\rho_1\|\gamma)\leq\beta_{x-\epsilon}(\rho_2\|\gamma)$
$\Updownarrow$	$\Uparrow$
$(\rho_1,\gamma)\succ_{(\epsilon,0)}(\rho_2,\gamma)$	$\rho_1 \xrightarrow[\text{TO}]{\epsilon} \rho_2$
$\Updownarrow$	$\Uparrow$
$\underset{x\in(\epsilon,1)}{\forall}\ \beta_x(\rho_1\|\gamma)\leq\beta_{x-\epsilon}(\rho_2\|\gamma)$	$\underset{x\in(\epsilon,1)}{\forall}\ \widetilde{\beta}_x(\rho_1\|\gamma)\leq\widetilde{\beta}_{x-\epsilon}(\rho_2\|\gamma)$

*Int. J. Theor. Phys. 39, 2717 (2000)

Formal statement of the problem

Optimal transformation rate $R_n^*(\epsilon_n)$ **is the largest rate** R_n **such that:**

$$\left(\rho_1^{\otimes n}, \sigma_1^{\otimes n}\right) \succeq_{(\epsilon_n, 0)} \left(\rho_2^{\otimes R_n n}, \sigma_2^{\otimes R_n n}\right).$$

Technical goal: Find the asymptotic scaling of $R_n^*(\epsilon_n)$ for $[\rho_2, \sigma_2] = 0$ in various error regimes.

As a result, we will also want to get optimal transformation rates for:

Thermodynamics

(choose $\sigma_1 = \sigma_2 = \gamma$)

$$\rho_1^{\otimes n} \xrightarrow[\text{TO}]{\epsilon} \rho_2^{R_n n}$$

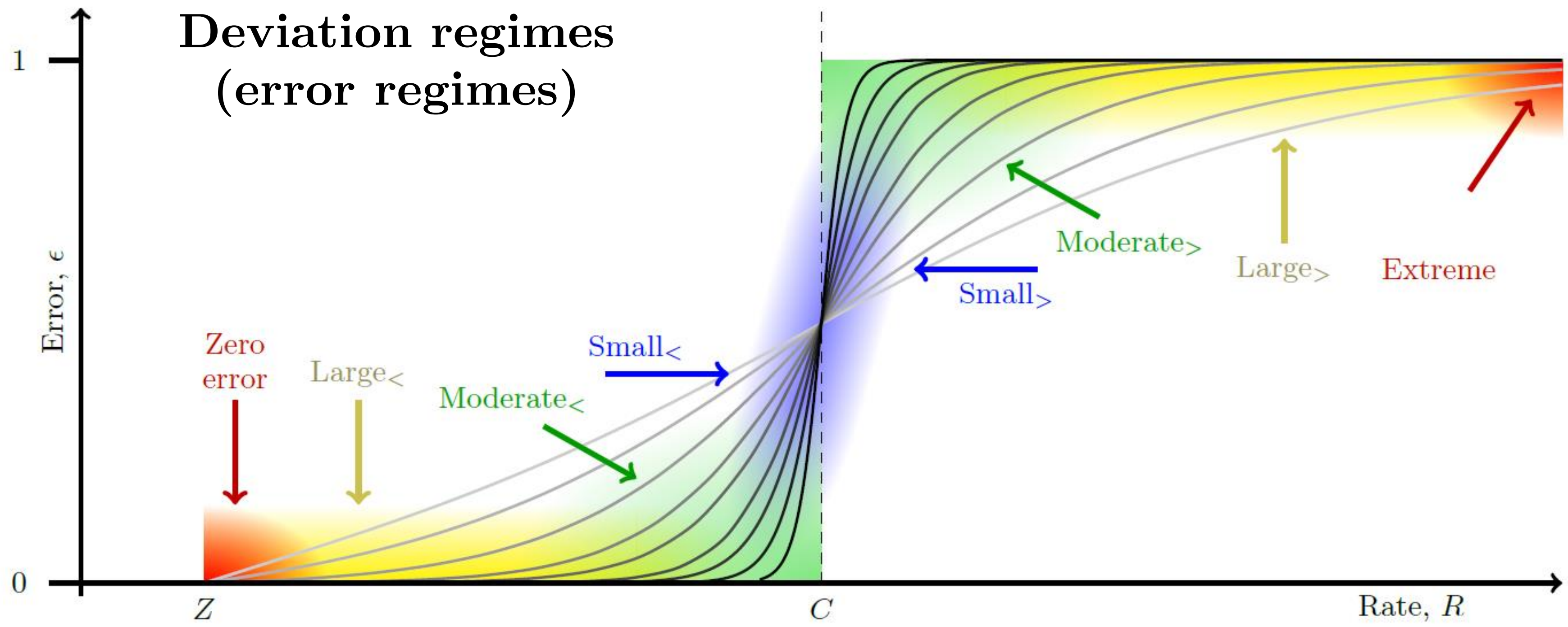
Entanglement

(choose $\rho_i = \mathrm{Tr}_B(|\psi_i^{AB}\rangle\langle\psi_i^{AB}|)$ and $\sigma_1 = \sigma_2 \propto \mathbb{I}$)

$$\left|\psi_1^{AB}\right\rangle^{\otimes n} \xrightarrow[\text{LOCC}]{\epsilon} \left|\psi_2^{AB}\right\rangle^{R_n n}$$

Results

Optimal transformation rates



Idea

Prove that up to second order asymptotics:

$$\widetilde{\beta}_x(\rho^{\otimes n}\|\sigma^{\otimes n}) \approx \beta_x(\rho^{\otimes n}\|\sigma^{\otimes n})$$

Regime	Error ϵ_n	Rate R_n	Tight?
Zero-error	Zero	Theorem 7	No
Large$_<$	Approaching 0 exponentially	Theorem 5	No
Moderate$_<$	Approaching 0 subexponentially	Theorem 4	Yes
Small$_<$	Constant, < 0.5	Theorem 3	Yes

Regime	Error ϵ_n	Rate R_n	Tight?
Small$_>$	Constant, > 0.5	Theorem 3	Yes
Moderate$_>$	Approaching 1 subexponentially	Theorem 4	Yes
Large$_>$	Approaching 1 exponentialy	Theorem 6	Yes
Extreme	Approaching 1 superexponentially	Theorem 8	Yes

*arXiv:2303.05524

Optimal transformation rates: small deviations

Optimal thermodynamic transformation rate from coherent to incoherent states

$$R_n^*(\epsilon) \simeq \frac{D(\rho_1\|\gamma) + \sqrt{V(\rho_1\|\gamma)/n} \cdot S_{1/\xi}^{-1}(\epsilon)}{D(\rho_2\|\gamma)}$$

Nonequilibrium free energy: $D(\rho\|\gamma) := (\mathrm{Tr}\rho\,(\log\rho - \log\gamma))$

Free energy fluctuations: $V(\rho\|\gamma) := \mathrm{Tr}\left(\rho\,(\log\rho - \log\gamma)^2\right) - D(\rho\|\gamma)^2$

Sesquinormal distribution*: $S_\nu^{-1}(\epsilon) = \inf_{x\in(\epsilon,1)} \sqrt{\nu}\Phi^{-1}(x) - \Phi^{-1}(x-\epsilon)$

Reversibility parameter: $\xi := \frac{V(\rho_1\|\gamma)}{D(\rho_1\|\gamma)} \Big/ \frac{V(\rho_2\|\gamma)}{D(\rho_2\|\gamma)}$

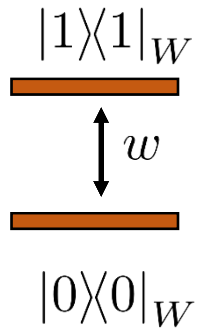
*arXiv:2303.05524

Optimal thermodynamic protocols

Work extraction

The amount of ε-deterministic work extractable from $\rho^{\otimes n}$ is the maximal value w such that:

$$\rho^{\otimes n} \otimes |0\rangle\langle 0|_W \xrightarrow[\text{TO}]{\epsilon} |1\rangle\langle 1|_W$$



Our results yield:

$$\frac{\beta w}{n} \simeq D(\rho\|\gamma) + \sqrt{\frac{V(\rho\|\gamma)}{n}}\Phi^{-1}(\epsilon)$$

Φ^{-1} : Inverse of the standard normal distribution

Information erasure

The amount of work needed to reset $\rho^{\otimes n}$ is the minimal value of w such that:

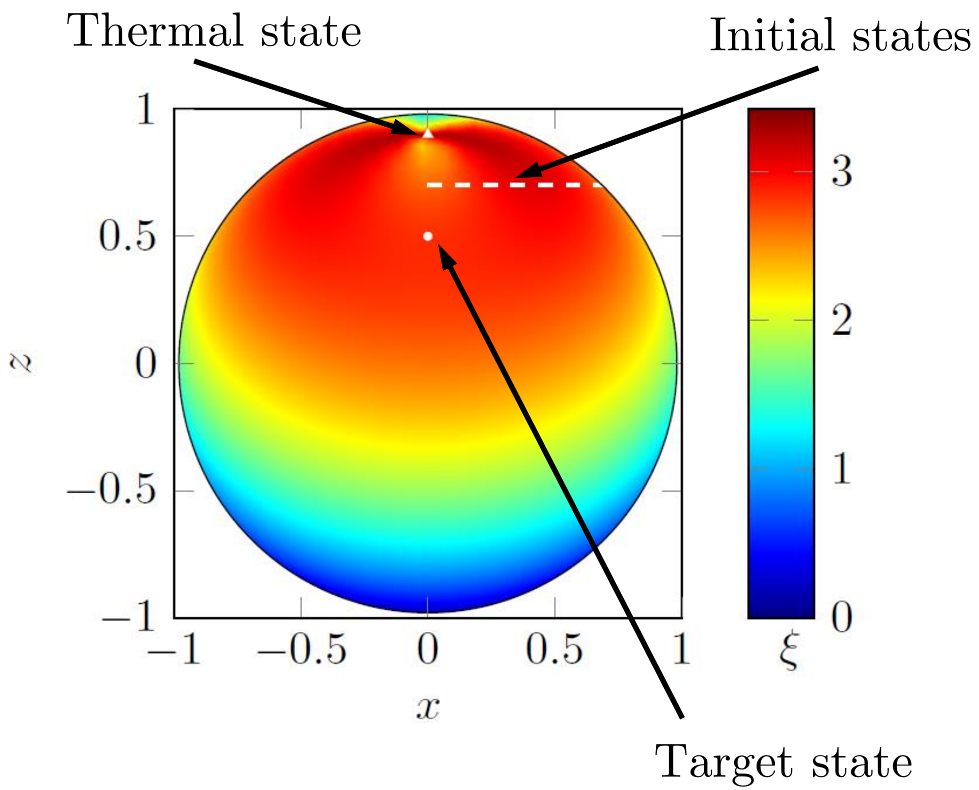
$$\rho^{\otimes n} \otimes |0\rangle\langle 0|_W \xrightarrow[\text{TO}]{\epsilon} |0\rangle\langle 0|^{\otimes n} \otimes |1\rangle\langle 1|_W$$

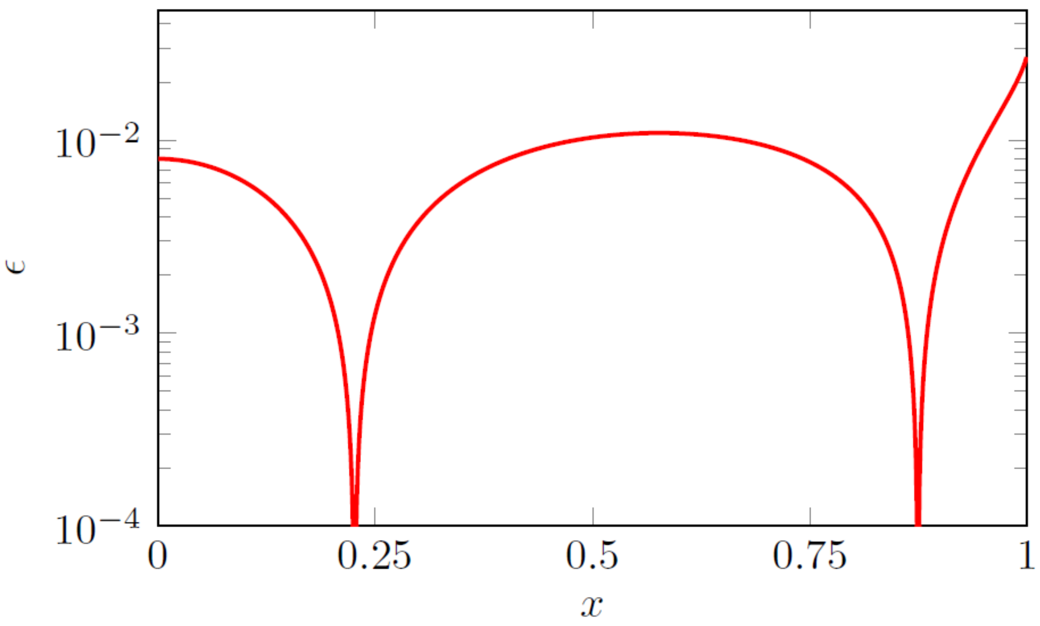
Our results yield:

$$\frac{\beta |w|}{n} \simeq S(\rho) - \sqrt{\frac{V(\rho\|\mathbb{I})}{n}}\Phi^{-1}(\epsilon)$$

Resource resonance

$$R_n^*(\epsilon) \simeq \frac{D(\rho_1\|\gamma) + \sqrt{V(\rho_1\|\gamma)/n} \cdot S^{-1}_{1/\xi}(\epsilon)}{D(\rho_2\|\gamma)} \quad \overset{\xi=1}{\Longrightarrow} \quad R_n^*(0) \simeq \frac{D(\rho_1\|\gamma)}{D(\rho_2\|\gamma)}$$





Outlook

- Use hypothesis testing approach to go beyond thermodynamic transformations of independent systems (include corelations).
- Investigate the explicit form of optimal thermodynamic protocols in different asymptotic regimes.
- Generalise our results so that they also apply to non-commutative output dichotomies.
- Generalise the obtained dichotomy results to multichotomies.
- Find the analogue of resonance phenomenon in a more traditional thermodynamic framework.
- Look for the resonance in other resource theories.

Check more:
arXiv:2303.05524 (2023)

Thank you!