RESOURCE ENGINES

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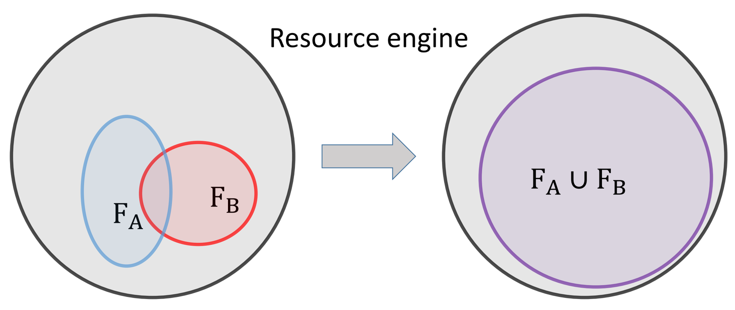
From a Heat to a Resource Engine

- In contrast to current approaches focused on scenarios with 1 set of constrained free operations (inspired by the access to a single heat bath), we propose to investigate the performance of *resource engines* [1], which generalise the concept of *heat engines* by replacing the access to 2 heat baths with 2 arbitrary constraints on state transformations.
- There are 2 agents (A, B), each of which is restricted to performing quantum operations from the subset ($\mathcal{F}_A$, $\mathcal{F}_B$) of free operations and preparing a subset (F_A, F_B) of free states.
- By fusing 2 resource theories described by $(F_A, \mathcal{F}_A)$ and $(F_B, \mathcal{F}_B)$, one can obtain a new one with free operations $\mathcal{F}_{AB} \supseteq \mathcal{F}_A \cup \mathcal{F}_B$ and free states $F_{AB} \supseteq F_A \cup F_B$.

Questions

- Can a resource engine defined by 2 constraints generate a full set of quantum operations?
- Can we achieve all possible final states starting from free states?



- Can we bound the number of strokes needed to obtain the above, and thus study the equivalents of engine's power and efficiency?
- What is achievable with just 3 strokes?

COHERENCE ENGINE

- A two-stroke engine fueled by the resource of quantum coherence.
- Free states – states diagonal in the distinguished bases (the theory is here restricted to pure state). As the set of operations we choose unitary transformations, and so the set of free operations is given by unitaries diagonal in the distinguished bases. More formally,

$$F_A = \{|i\rangle\}_{i=1}^d, \qquad F_B = \{U^\dagger |i\rangle\}_{i=1}^d,$$

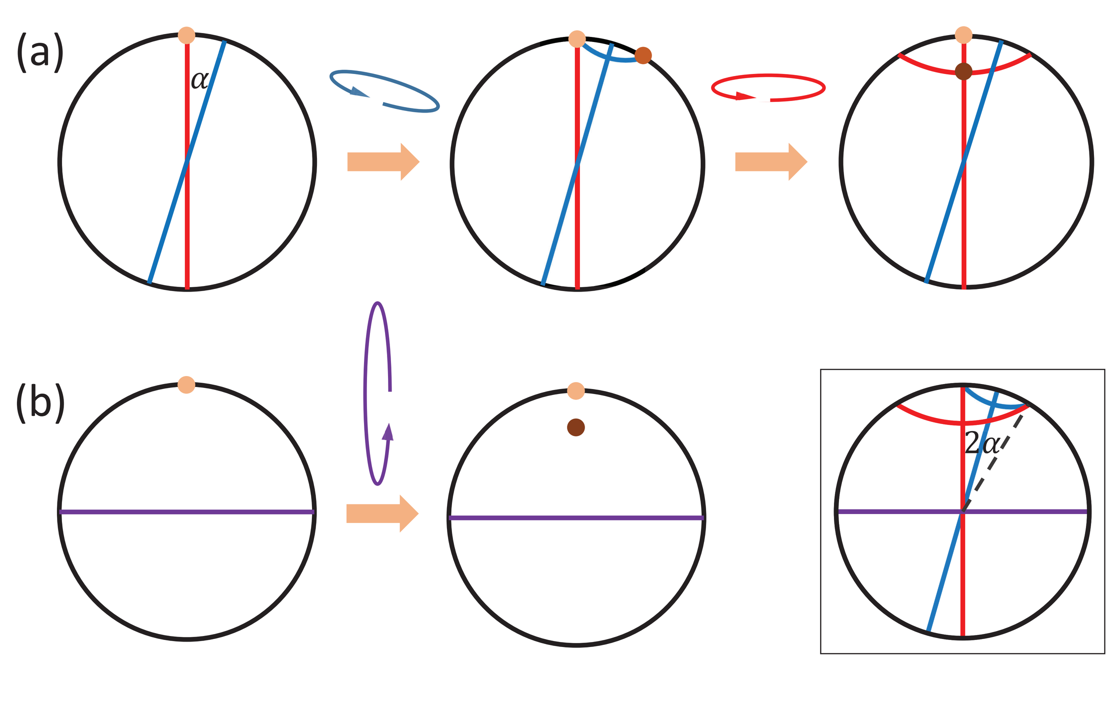
where $U \in \mathcal{U}_d(\mathbb{C})$ is describing the relative orientation of the 2 bases, and

$$\mathcal{F}_A = \mathcal{DU}_d(\mathbb{C}) = \left\{ \sum_{i=1}^d u_{ii} |i\rangle\langle i| : |u_{ii}| = 1 \right\},$$

$$\mathcal{F}_B = \{U^\dagger D U : \ D \in \mathcal{DU}_d(\mathbb{C})\}.$$

Elementary qubit example

- F_A and F_B can be seen as the σ_z and the $\cos(\alpha)\sigma_z + \sin(\alpha)\sigma_x$ eigenbases, while $\mathcal{F}_A$ and $\mathcal{F}_B$ – as rotations about real three-dimensional unit vectors $\hat{z}$ and $\hat{v}$, respectively.
- If $\hat{v} = \hat{z}_\perp$ ($\alpha = \pi/2$), then, by the Euler angles decomposition, only 3 strokes are needed to generate an arbitrary unitary transformation. Whenever $\hat{v} \neq \hat{z}_\perp$ ($\alpha < \pi/2$), the number of strokes increases to $\lceil \pi/\alpha \rceil + 1$ (cf., e.g., [2]).



Proposition 1. *A and B can, by means of a resource engine, generate any unitary matrix of order 2, with the minimal number of strokes equal to $\lceil \pi/\alpha \rceil + 1$.*

- When an optimal state is achievable with just 3 strokes?
- By optimal we mean a state that is maximally resourceful for both parties, here a *mutually coherent* state [3], that is

$$\frac{1}{\sqrt{2}}\left(e^{i\alpha_0}|0\rangle + e^{i\alpha_1}|1\rangle\right) = \frac{1}{\sqrt{2}}\left(e^{i\beta_0}U^\dagger|0\rangle + e^{i\beta_1}U^\dagger|1\rangle\right).$$

Proposition 2. *If U in the definition of $\mathcal{F}_B$ is s.t.*

$$U = e^{i\phi}\begin{bmatrix} e^{i\varphi_0}\cos(\varphi) & -e^{-i\varphi_1}\sin(\varphi) \\ e^{i\varphi_1}\sin(\varphi) & e^{-i\varphi_0}\cos(\varphi) \end{bmatrix}, \quad \varphi \in \left[\frac{\pi}{8}, \frac{3\pi}{8}\right],$$

then A and B can generate a mutually coherent state with only 3 strokes of their resource engine.

Condition on getting all operations

(H1) There exist $N \in \mathbb{N}$ and $D_1, \ldots, D_{2N} \in \mathcal{DU}_d(\mathbb{C})$ s.t. $D_1(U^\dagger D_2 U) D_3 (U^\dagger D_4 U) \ldots D_{2N-1}(U^\dagger D_{2N} U)$ is a matrix with all non-zero entries.

(H2) There exists $N \in \mathbb{N}$ s.t. $(P_U^T P_U)^N$, with $p_{ij} = 0$ for $u_{ij} = 0$ and $p_{ij} = 1$ otherwise, has all non-zero entries.

Theorem 1. *If U in the definition of $\mathcal{F}_B$ satisfies either (H1) or (H2), then any unitary matrix can be written as a product comprised of N matrices from $\mathcal{F}_A$ and N from $\mathcal{F}_B$.*

Proof. In the proof we use the fact that any subgroup $\mathcal{V}$ in $\mathcal{U}_d(\mathbb{C})$ containing $\mathcal{DU}_d(\mathbb{C})$ and a matrix $(w_{ij})_{i,j=1}^d$ such that $w_{ij} \neq 0$ for any $1 \leq i, j \leq d$ is a full unitary group [4]. □

Bounds on the number of strokes to get all operations

Theorem 2 (LOWER BOUND). *The number of resource engine's strokes needed to generate any unitary matrix of order d is lower bounded by*

$$\frac{2\log(d-1)}{\log((d-2)c_U + 1)} \ \text{for } d \in \mathbb{N}\backslash\{1,2\} \quad \text{and} \quad \frac{2}{c_U} \ \text{for } d = 2,$$

where $c_U := \max_{a \neq b} \sum_{j=1}^d |\bar{u}_{j,a}||u_{j,b}| = \max_{a\neq b}\langle a|(X_U^T X_U)|b\rangle$ with $x_{i,j} = |u_{i,j}|$.

Theorem 3 (UPPER BOUND). *Let M denote the number of resource engine's strokes needed to generate the Fourier matrix of order d. Then the upper bound on the number of strokes needed to generate any unitary operation is equal to $2M(d-1) + 2(d-1) + 1$.*

Proof. We utilize the following 2 facts: (1) any generic matrix can be written as a product of circulant and diagonal matrices with the number of factors being $2d-1$ at most, (2) any circulant matrix can be written as a product of a Fourier transform, a diagonal matrix and an inverse Fourier transform [5]. □

Condition on getting the optimal state

Theorem 4. *Let $U = (u_{i,j})_{i,j=1}^d$ define the set $\mathcal{F}_B$. The necessary conditions for A and B to produce a mutually coherent state with just two resource engine's strokes are:*

$$\exists_{1\leq l\leq d} \forall_{1\leq m\leq d} \max_{1\leq i\leq d} |u_{m,i}\bar{u}_{l,i}| \leq \frac{1}{2}\left(\sum_{j=1}^d |u_{m,j}\bar{u}_{l,j}| + \frac{1}{\sqrt{d}}\right).$$

ATHERMALITY ENGINE

- A two-stroke heat engine, but with agents not allowed to use any ancillary systems.
- Interconversion problem: *which final states are achievable from a given initial state via thermal operations?* We consider the quasi-classical version of the problem with initial and final states of the system commuting with the system's thermal Gibbs state γ [the existence of a thermal operation mapping $\boldsymbol{p}$ to $\boldsymbol{q}$ is then fully characterized by a thermomajorization condition $\boldsymbol{p} \succ_\gamma \boldsymbol{q}$ [6]].
- $\mathcal{F}_A$ ($\mathcal{F}_B$) – thermal operations with inverse temperature α (β, $\beta < \alpha$) and the corresponding free state $\boldsymbol{\gamma}$ ($\boldsymbol{\Gamma}$) s.t.

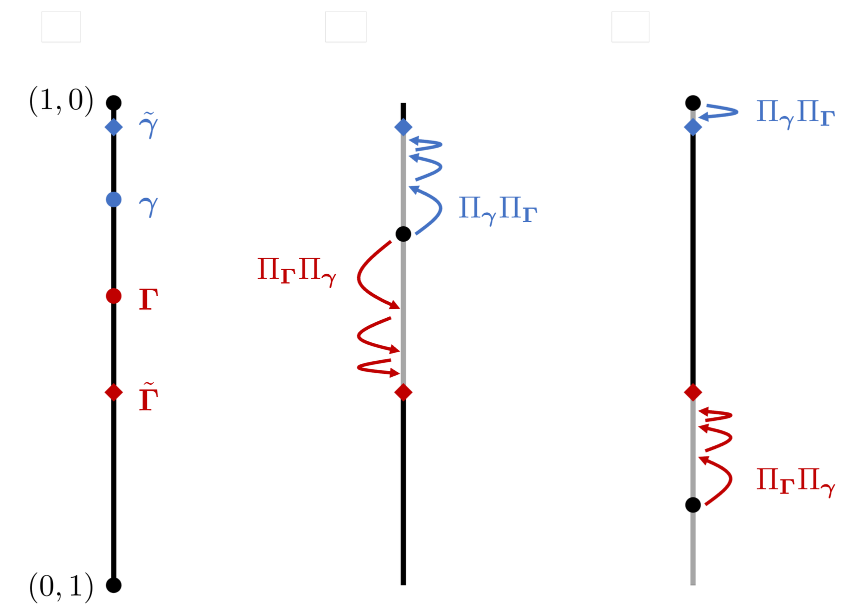
$$\gamma_k = \frac{e^{-\alpha E_k}}{\sum_{i=1}^d e^{-\alpha E_i}}, \qquad \left(\Gamma_k = \frac{e^{-\beta E_k}}{\sum_{i=1}^d e^{-\beta E_i}}\right),$$

where $\{E_i\}$ denotes the energy levels of the system.

- $\boldsymbol{p}^{(N-1)}$ that can be obtained via a sequence of thermomajorisations either from $\boldsymbol{\gamma}$ or $\boldsymbol{\Gamma}$, i.e.

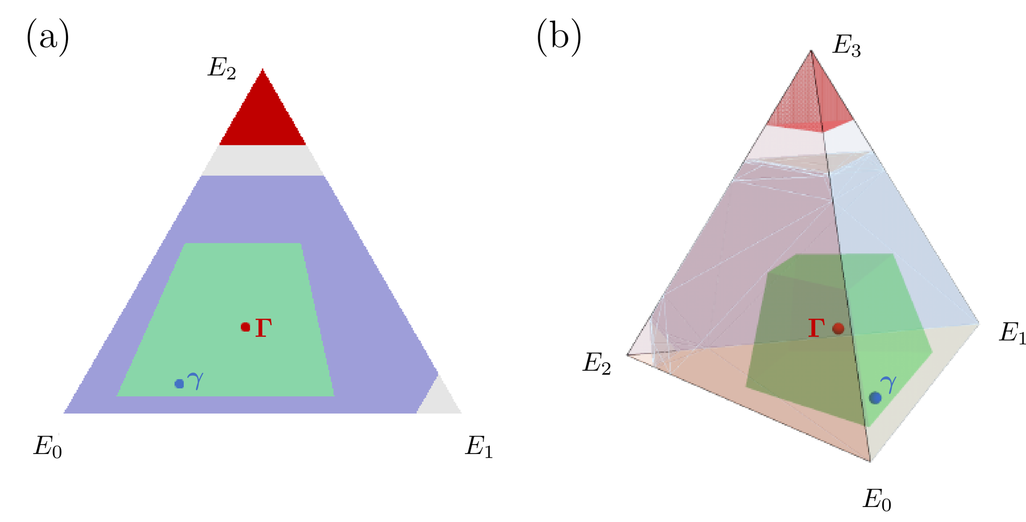
$$\boldsymbol{p}^{(0)} = \boldsymbol{\gamma} \succ_\Gamma \boldsymbol{p}^{(1)} \succ_\gamma \boldsymbol{p}^{(2)} \succ_\Gamma \boldsymbol{p}^{(3)} \succ_\gamma \ldots \boldsymbol{p}^{(N-1)},$$
$$\boldsymbol{p}^{(0)} = \boldsymbol{\Gamma} \succ_\gamma \boldsymbol{p}^{(1)} \succ_\Gamma \boldsymbol{p}^{(2)} \succ_\gamma \boldsymbol{p}^{(3)} \succ_\Gamma \ldots \boldsymbol{p}^{(N-1)}.$$

Elementary qubit example



Bounding the set of achievable qudit states

Theorem 5. *The set F_{AB} of free states arising from fusing 2 resource theories of thermodynamics with 2 different temperatures is bounded as follows: $\forall \boldsymbol{p} \in F_{AB} \quad p_d \leq \frac{\Gamma_d}{\Gamma_{d-1}}$, where $\boldsymbol{\Gamma}$ denotes the thermal state corresponding to higher temperature.*



Full set of achievable states for $\beta = 0$

Theorem 6. *The set F_{AB} of free states arising from fusing two resource theories of thermodynamics, one with finite and the other with infinite temperature, is given by the full probability simplex. Moreover, the convergence to the full simplex with the number of strokes N is exponential.*

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Acknowledgements

This work was supported by:

- Team-Net programme of the Foundation for Polish Science, co-financed by the European Union under the European Regional Development Fund; grant number POIR.04.04.00-00-17C1/18-00;
- SONATA BIS programme of National Science Centre Poland; grant number 2016/22/E/ST6/00062.

Fundacja na rzecz Nauki Polskiej
Fundusze Europejskie Wiedza Edukacja Rozwój
Rzeczpospolita Polska
Unia Europejska Europejski Fundusz Społeczny