FDR for thermodynamic distillation processes

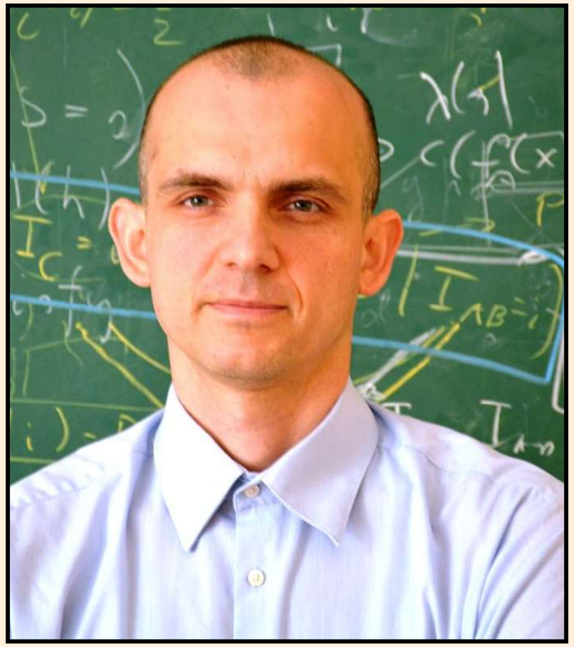
Alexssandre de Oliveira Junior

Faculty of Physics, Astronomy and Applied Computer Science,
Jagiellonian University

Quantum Chaos and Quantum Information
March 15, 2021

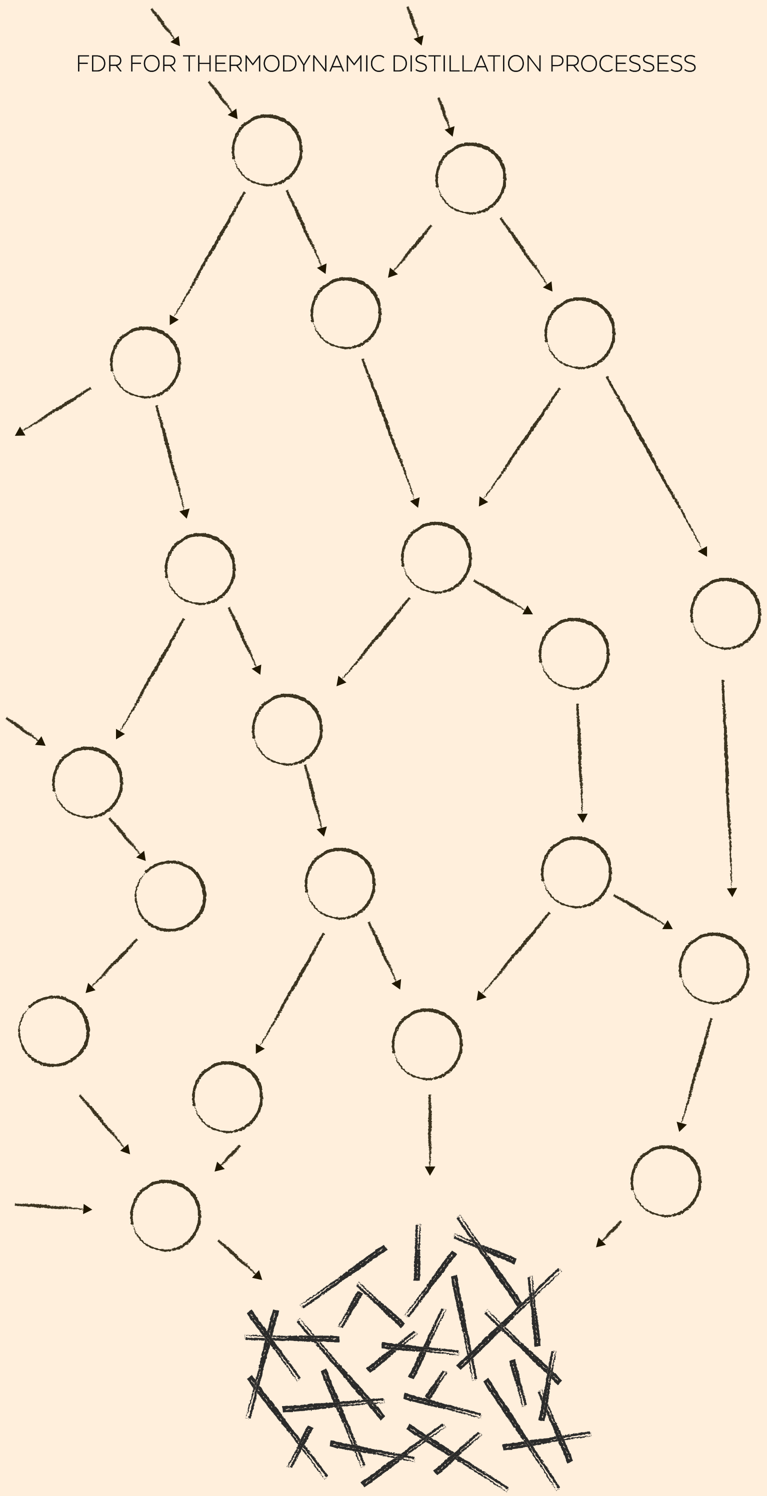
Collaborators

Kamil Korzekwa
Jagiellonian University, Krakow



Michał Horodecki
ICTQT, Gdansk

Tanmoy Biswas
ICTQT, Gdansk



Outline

I. Introduction

II. Resource theory of thermodynamics

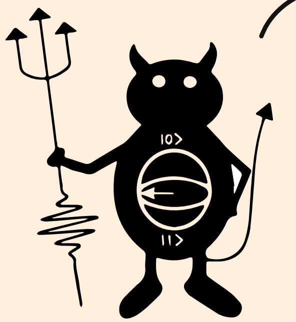
III. Results

IV. Applications

V. Outlook

Introduction

Thermodynamics in a nutshel



A fundamental question in thermodynamics

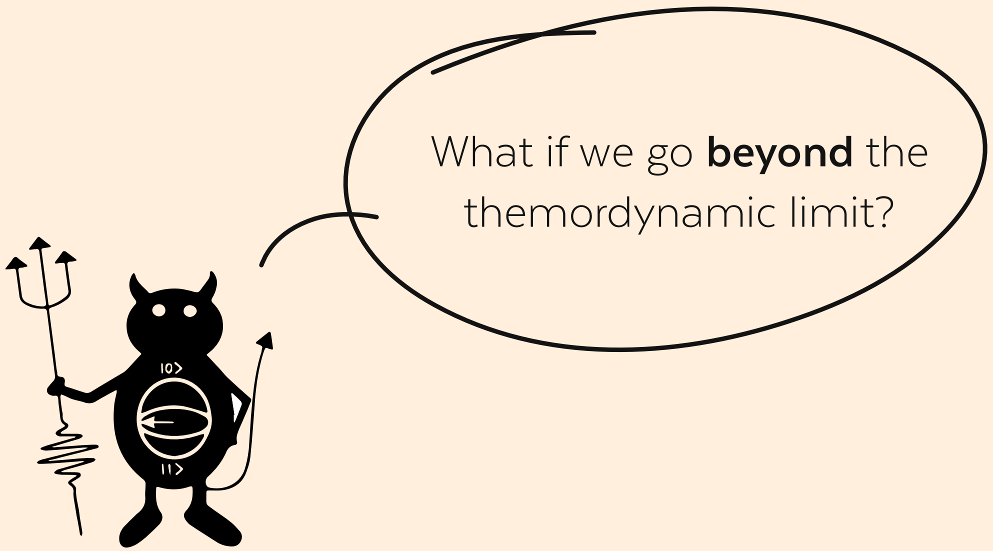
$$X \overset{?}{\to} Y$$

Standard thermodynamics

- Laws of thermodynamics: $W \rightleftharpoons Q$
- State variables: $X = (\underbrace{P, V, T}), Y = (P', V', T')$

  statistical nature ⟶ well-defined
- Thermodynamic limit: $N \to \infty$, $\tau_s \to \infty$

Thermodynamics in a nutshel

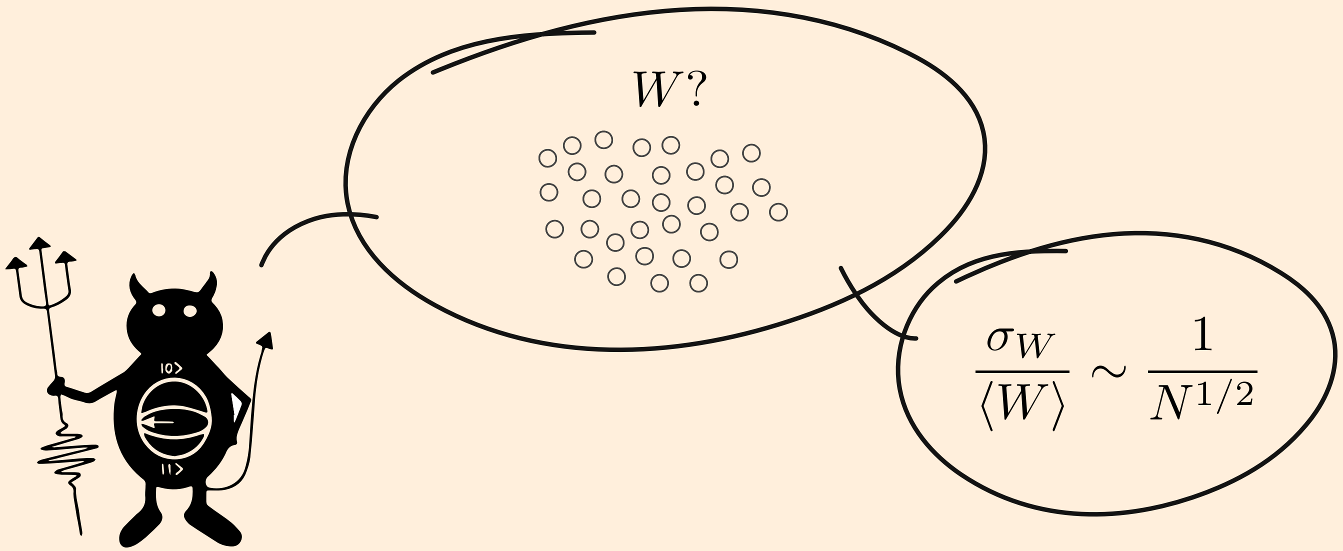


Standard thermodynamics

- Laws of thermodynamics: $W \rightleftharpoons Q$
- State variables: $X = (P, V, T), Y = (P', V', T')$

  statistical nature → ~~well-defined~~
- Thermodynamic limit: $N \not\rightarrow \infty$, $\tau_s \not\rightarrow \infty$

Thermodynamics in a nutshel



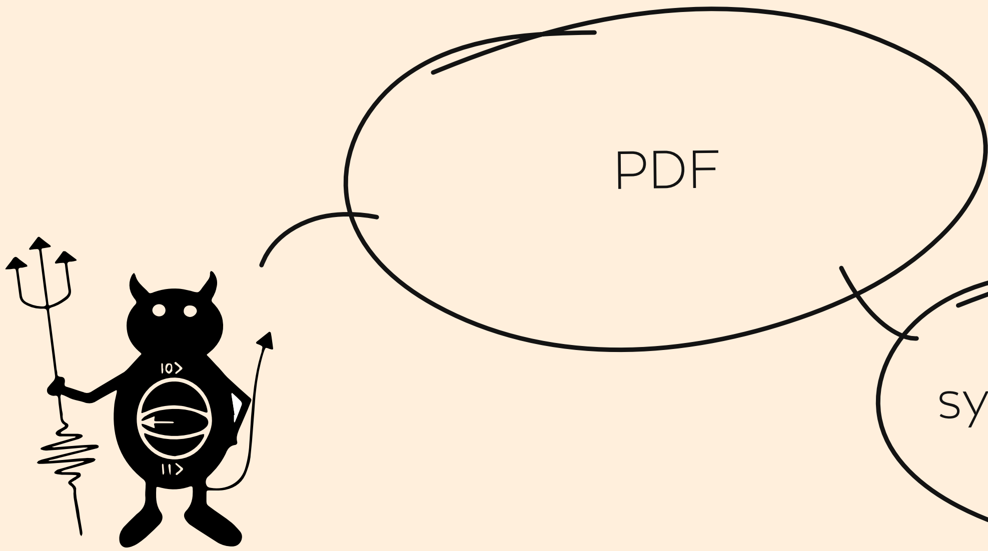
Standard thermodynamics

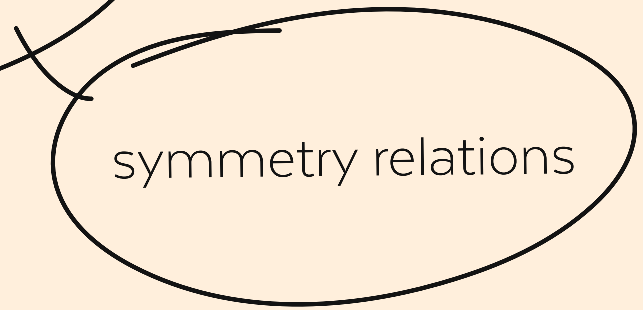
- Laws of thermodynamics: $W \rightleftharpoons Q$
- State variables
- Thermodynamic limit

Non.equilibrium thermodynamics

- Fluctuations!
- Stochastic variables
- Fluctuation-theorems

Thermodynamics in a nutshel





Standard thermodynamics

- Laws of thermodynamics: $W \rightleftharpoons Q$
- State variables
- Thermodynamic limit

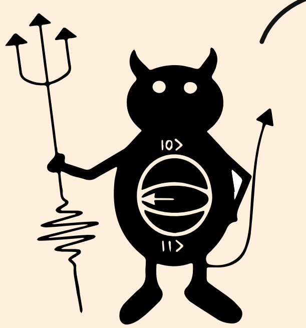
Non.equilibrium thermodynamics

- Fluctuations!
- Stochastic variables
- Fluctuation-theorems

Thermodynamics in a nutshel

$$e^{-\beta \Delta F} = \langle e^{-\beta W} \rangle$$

Jarzynski equality



Standard thermodynamics

- Laws of thermodynamics: $W \rightleftharpoons Q$
- State variables
- Thermodynamic limit

Non.equilibrium thermodynamics

- Fluctuations!
- Stochastic variables
- Fluctuation-theorems

Thermodynamics in a nutshel

Standard thermodynamics

- Laws of thermodynamics: $W \rightleftharpoons Q$
- State variables
- Thermodynamic limit

Non.equilibrium thermodynamics

- Fluctuations!
- Stochastic variables
- Fluctuation-theorems

Quantum thermodynamics

- Quantum features
- Information-theoretic nature
- Restrictions?

Thermodynamics in a nutshel

our work

Standard thermodynamics

- Laws of thermodynamics: $W \rightleftharpoons Q$
- State variables
- Thermodynamic limit

Non.equilibrium thermodynamics

- Fluctuations!
- Stochastic variables
- Fluctuation-theorems

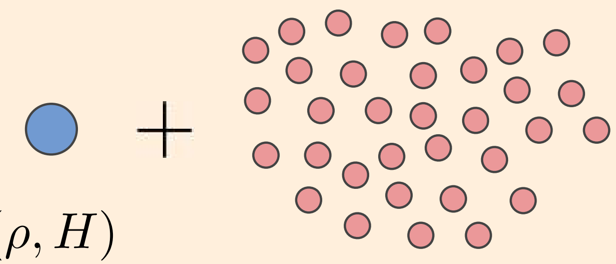
Quantum thermodynamics

- Quantum features
- Information-theoretic nature
- Restrictions?

Resource theories of Thermodynamics

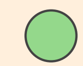
Resource theory of thermodynamics

Indentifying the set of **thermodynamically-free states**



(ρ, H) (γ_E, H_E)

$$\Longrightarrow \quad \gamma = \frac{e^{-\beta H}}{Z} \quad , \quad Z = \mathrm{Tr}(e^{-\beta H}) \quad ,$$

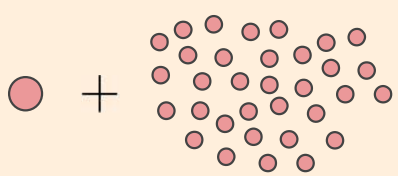


Thermodynamic transformations are modelled by **thermal operations**

$$\mathcal{E}(\rho) = \mathrm{Tr}_E(U(\rho \otimes \gamma_E)U^\dagger) \quad \text{with} \quad [U, H \otimes \mathbb{1}_E + \mathbb{1}_E \otimes H_E] = 0$$

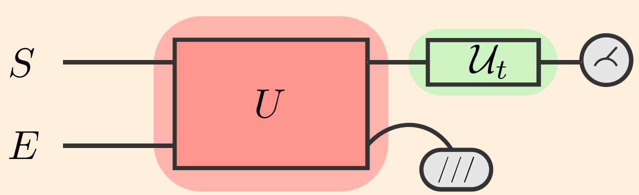
Energy-conserving interaction

i. $\mathcal{E}(\gamma) = \gamma$

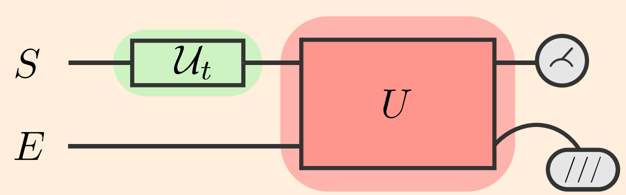


$\mathcal{E}(\bullet) = \bullet \longrightarrow$ **2nd law**

ii. $\mathcal{E} \circ \mathcal{U}_t = \mathcal{U}_t \circ \mathcal{E}$



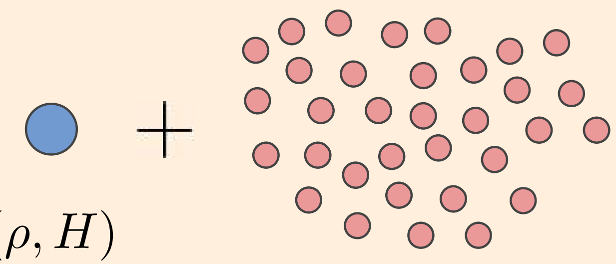
$=$



$\longrightarrow$ **1st law**

Resource theory of thermodynamics

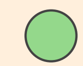
Indentifying the set of **thermodynamically-free states**



(ρ, H)

(γ_E, H_E)

$$\Longrightarrow \quad \gamma = \frac{e^{-\beta H}}{Z} \quad , \quad Z = \mathrm{Tr}(e^{-\beta H}) \quad ,$$

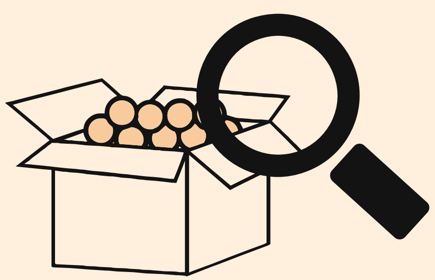


Thermodynamic transformations are modelled by **thermal operations**

$$\mathcal{E}(\rho) = \mathrm{Tr}_E(U(\rho \otimes \gamma_E)U^\dagger) \quad \text{with} \quad [U, H \otimes \mathbb{1}_E + \mathbb{1}_E \otimes H_E] = 0$$

Energy-conserving interaction

Thermodynamic **monotone** $\phi : \mathcal{S}_d \to \mathbb{R}_+ \cup \{0\}$



i. $\phi(\mathcal{E}(\rho)) \leq \phi(\rho)$

ii. $\phi(\gamma) = 0$

$\longrightarrow$ $D(\rho\|\gamma) = \mathrm{Tr}(\rho(\log\rho - \log\gamma))$

Information + thermo

Generalised **free energy**

Information-theoretic intermission

Expression	Interpretation
$D(\rho\\|\gamma) = \mathrm{Tr}(\rho(\log\rho - \log\gamma))$	$\beta\left[\underbrace{\left(\mathrm{tr}(\rho H) - \frac{S(\rho)}{\beta}\right)}_{\text{Free energy}} - \underbrace{\left(-\frac{\log Z}{\beta}\right)}_{\text{Free energy of } \gamma}\right]$
$V(\rho\\|\gamma) = \mathrm{Tr}\left(\rho\left(\log\rho - \log\gamma - D(\rho\\|\gamma)\right)^2\right)$	Fluctuations of a given random variable

Information-theoretic intermission

Expression	Interpretation
$D(\rho\|\gamma) = \mathrm{Tr}(\rho(\log\rho - \log\gamma))$	$\beta\left[\underbrace{\left(\mathrm{tr}(\rho H) - \frac{S(\rho)}{\beta}\right)}_{\text{Free energy}} - \underbrace{\left(-\frac{\log Z}{\beta}\right)}_{\text{Free energy of } \gamma}\right]$
$V(\rho\|\gamma) = \mathrm{Tr}\left(\rho\left(\log\rho - \log\gamma - D(\rho\|\gamma)\right)^2\right)$	$V(\psi\|\gamma) = \langle E^2\rangle - \langle E\rangle^2$

Information-theoretic intermission

Expression	Interpretation
$D(\rho\|\gamma) = \mathrm{Tr}(\rho(\log\rho - \log\gamma))$	$\beta\left[\underbrace{\left(\mathrm{tr}(\rho H) - \frac{S(\rho)}{\beta}\right)}_{\text{Free energy}} - \underbrace{\left(-\frac{\log Z}{\beta}\right)}_{\text{Free energy of } \gamma}\right]$
$V(\rho\|\gamma) = \mathrm{Tr}\left(\rho\left(\log\rho - \log\gamma - D(\rho\|\gamma)\right)^2\right)$	$V(\gamma'\|\gamma) = \underbrace{\frac{\partial\langle E\rangle_{\gamma'}}{\partial T'}}_{\text{Specific heat capacity}} \underbrace{\left(1 - \frac{T'}{T}\right)^2}_{\text{Carnot factor}}$

Information-theoretic intermission

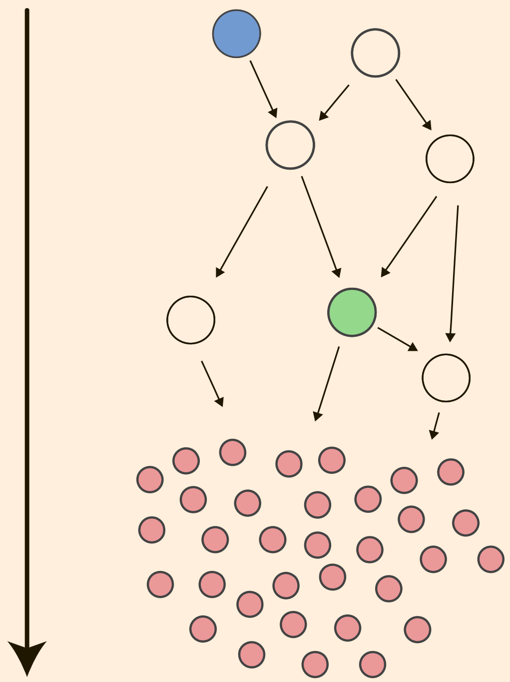
Expression	Interpretation
$D(\rho\Vert\gamma) = \mathrm{Tr}(\rho(\log\rho - \log\gamma))$	$\beta\left[\underbrace{\left(\mathrm{tr}(\rho H) - \frac{S(\rho)}{\beta}\right)}_{\text{Free energy}} - \underbrace{\left(-\frac{\log Z}{\beta}\right)}_{\text{Free energy of }\gamma}\right]$
$V(\rho\Vert\gamma) = \mathrm{Tr}\left(\rho\left(\log\rho - \log\gamma - D(\rho\Vert\gamma)\right)^2\right)$	$V(\gamma'\Vert\gamma) = \underbrace{\frac{\partial\langle E\rangle_{\gamma'}}{\partial T'}}_{\text{Specific heat capacity}}\underbrace{\left(1 - \frac{T'}{T}\right)^2}_{\text{Carnot factor}}$
$W(\rho\Vert\gamma) := \mathrm{Tr}\left(\rho\left(\frac{\log\rho - \log\gamma - D(\rho\Vert\gamma)}{\sqrt{V(\rho\Vert\gamma)}}\right)^3\right)$	$W(\gamma'\Vert\gamma) = -\sqrt{\frac{k_B}{{c_T^3}'}}\left(T'\frac{\partial {c_T}'}{\partial T'} + 2{c_T}'\right)$

Resource theory of thermodynamics

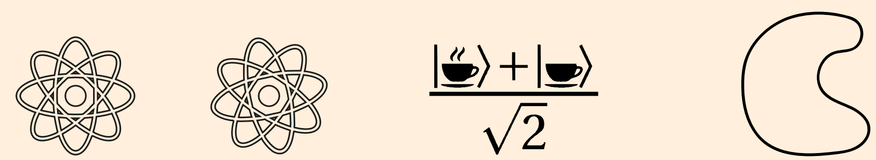
General **interconversion** problem: for initial state ρ, target state σ, thermal bath β $\Longrightarrow$ $\mathcal{E}(\rho) = \sigma$

out of equilibrium

equilibrium



Recently developed resource theories



Entanglement	Coherence	Asymmetry
Non-local	Coherent	Asymetric
↓	↓	↓
Separable	Incoherent	Symmetric

Resource theory of thermodynamics

General **interconversion** problem: for initial state ρ, target state σ, thermal bath β $\Longrightarrow$ $\mathcal{E}(\rho) = \sigma$

! General answer not known beyond the simplest qubit case

Phys. Rev. X 5, 021001 (2015)
Nat.Commun. 6, 7689 (2015)

! For **energy-incoherent states** the set of necessary and sufficient conditions was found

Nat.Commun. 4, 2059 (2013)

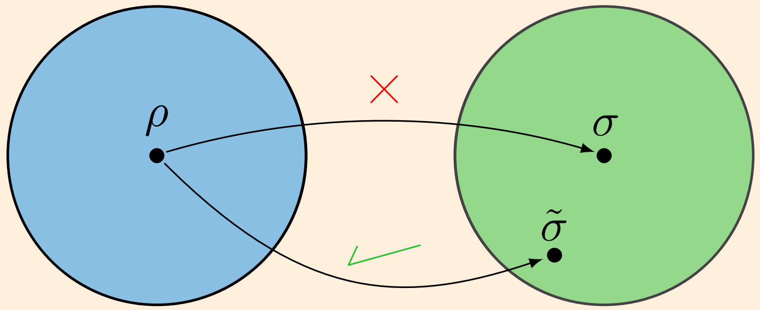
$[\rho, H] = [\sigma, H] = 0 \implies$ states represented by: $\mathbf{p} = \mathrm{eig}(\rho), \mathbf{q} = \mathrm{eig}(\sigma)$

Returning to the question...

$$\mathcal{E}(\rho) = \sigma : \mathbf{p} \succ^{\beta} \mathbf{q}$$

Resource theory of thermodynamics

General **interconversion** problem: for initial state ρ, target state σ, thermal bath β $\Longrightarrow$ $\mathcal{E}(\rho) = \sigma$



final state

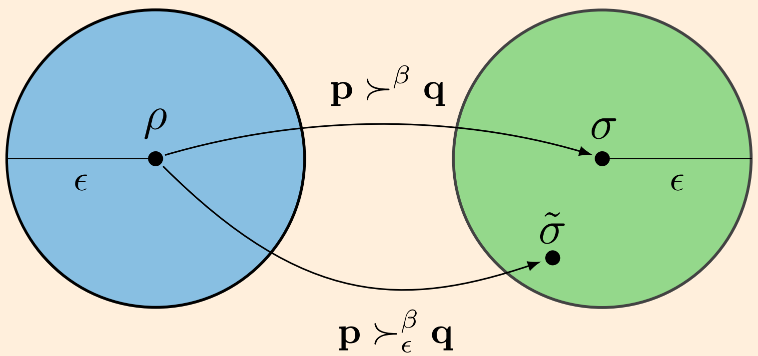
ϵ - approximate **interconversion** problem: for initial state ρ, target state σ, thermal bath β $\Longrightarrow$ $\mathcal{E}(\rho) = \tilde{\sigma}$

$\sigma \approx_\epsilon \tilde{\sigma}$ means $1 - F(\sigma, \tilde{\sigma}) \leq \epsilon$ with fidelity $F(\sigma, \tilde{\sigma}) = \left(\mathrm{Tr}\sqrt{\sqrt{\sigma}\tilde{\sigma}\sqrt{\sigma}}\right)$

$$\mathbf{p} \succ_\epsilon^\beta \mathbf{q}$$

Resource theory of thermodynamics

! Approximate interconversion problem with **finite** system: $\mathcal{E}(\rho^{\otimes N}) = \tilde{\sigma}^{\otimes M}$



Quantum, vol. 2, p.108, 2018

! Second order rate for energy-incoherent states

! Thermodynamic irreversibility (rigorously)

! Optimal values of distillable work and work of formation

Not answered

? For general states (not only energy-incoherent)

? Going beyond the second-order asymptotic state interconversion, i.e., rates for any N

? Have only one battery system instead of N

Results

Thermodynamic distillation process

An ϵ-approximate **thermodynamic** distillation process from an initial to a target state

$$(\rho, H) \xrightarrow{\mathcal{E}} (\tilde{\rho}, \tilde{H})$$

Eigenstate of $|\tilde{E}^{(n)}_{k_n}\rangle$ correspoding to energy $\tilde{E}^{(n)}_{k}$

where $\tilde{\rho} = \bigotimes_{m=1}^{\tilde{N}} |\tilde{E}^{(n)}_{k_n}\rangle\langle\tilde{E}^{(n)}_{k_n}|$

ϵ away from $\tilde{\rho}$ in the infidelity distance

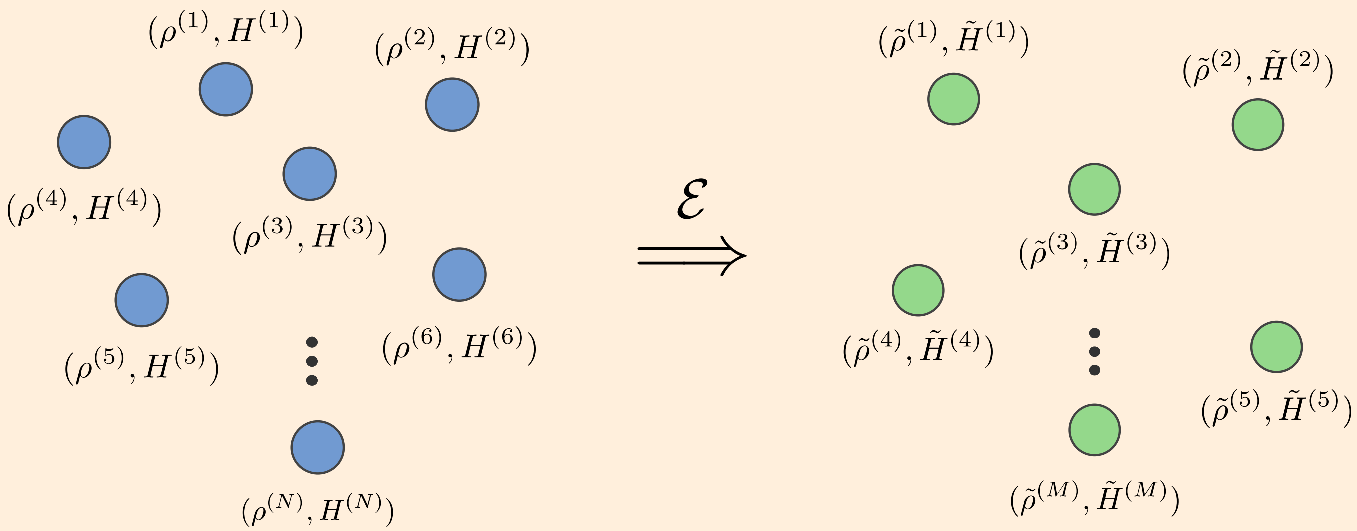
$$\delta(\rho_1, \rho_2) := 1 - \left(\mathrm{Tr}\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}\right)^2$$

Dissipated free energy rescaled by its **fluctuations**

$$\frac{W^{\text{diss}}}{\sigma} := \frac{D(\rho\|\gamma) - D(\tilde{\rho}\|\tilde{\gamma})}{\sqrt{V(\rho\|\gamma)}}$$

FDR for incoherent states

Theorem 1. Fluctuation-dissipation relation for incoherent states

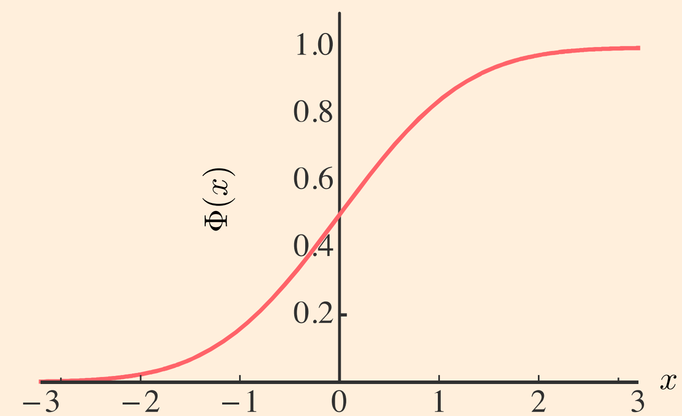


$$H = \sum_{n=1}^{N} H^{(n)} \quad , \quad \rho = \bigotimes_{n=1}^{N} \rho^{(n)} \qquad \qquad \tilde{H} = \sum_{n=1}^{\tilde{N}} \tilde{H}^{(n)} \quad , \quad \tilde{\rho} = \bigotimes_{n=1}^{\tilde{N}} |\tilde{E}_{k_n}^{(n)}\rangle\langle\tilde{E}_{k_n}^{(n)}|$$

$$N \to \infty$$

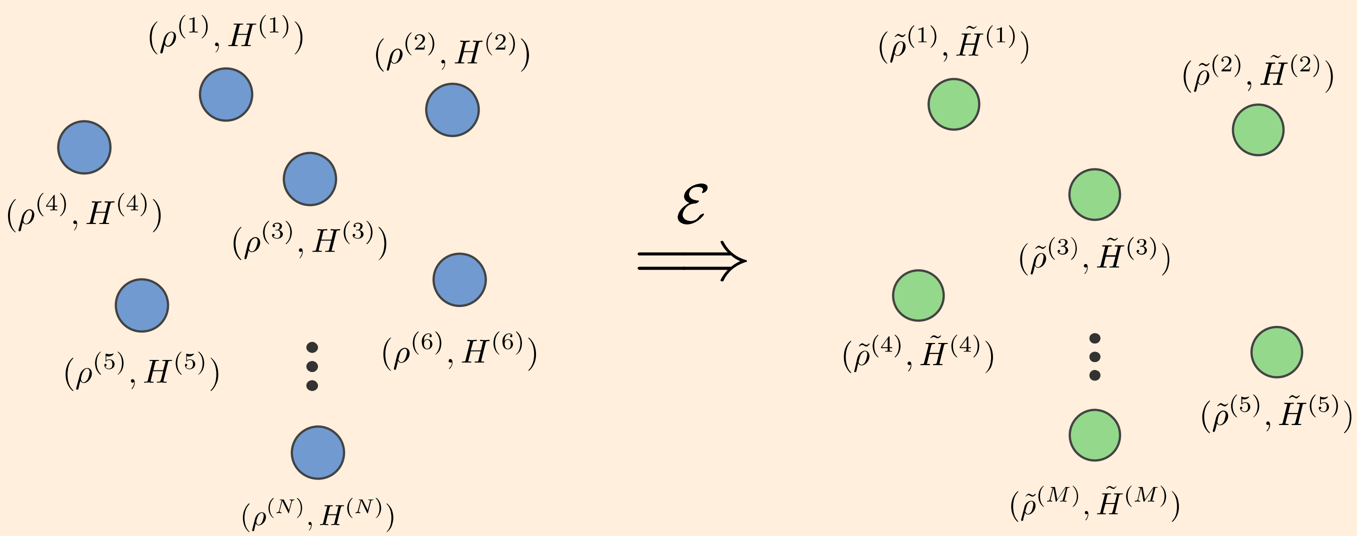
$$\epsilon \simeq 1 - \Phi\left(\frac{W^{\text{diss}}}{\sigma}\right)$$

Cumulative normal distribution



FDR for incoherent states

Theorem 1. Fluctuation-dissipation relation for incoherent states



$$H = \sum_{n=1}^{N} H^{(n)} \quad , \quad \rho = \bigotimes_{n=1}^{N} \rho^{(n)}$$

$$\tilde{H} = \sum_{n=1}^{\tilde{N}} \tilde{H}^{(n)} \quad , \quad \tilde{\rho} = \bigotimes_{n=1}^{\tilde{N}} |\tilde{E}^{(n)}_{k_n}\rangle\langle\tilde{E}^{(n)}_{k_n}|$$

Berry-Esseen constant

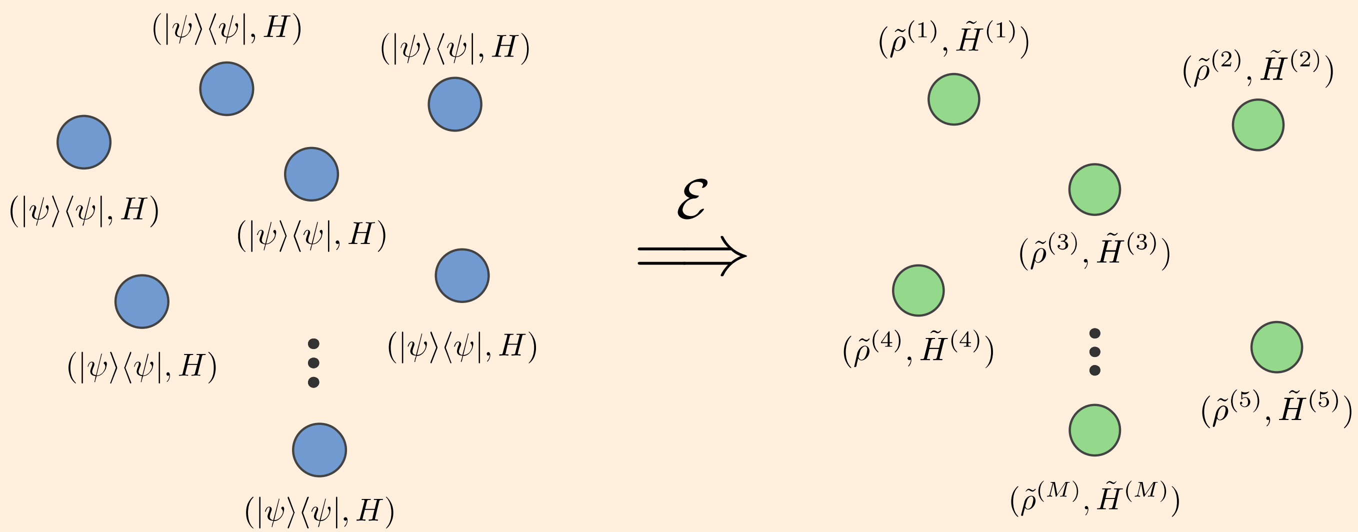
$$0.4097 \leq C \leq 0.4748$$

$$\epsilon \leq 1 - \Phi\left(\frac{W^{\text{diss}}}{\sigma}\right) + \frac{C\,W(\rho\|\gamma)}{\sigma^3}$$

- **Beyond** the i.i.d case:
- Guarantees a transformation error for a **finite** N
- Fluctuation-dissipation relation!

FDR for i.i.d pure states

Theorem 2. Fluctuation-dissipation relation for i.i.d pure states



$$\rho = \bigotimes_{n=1}^{N} |\psi\rangle\langle\psi|$$

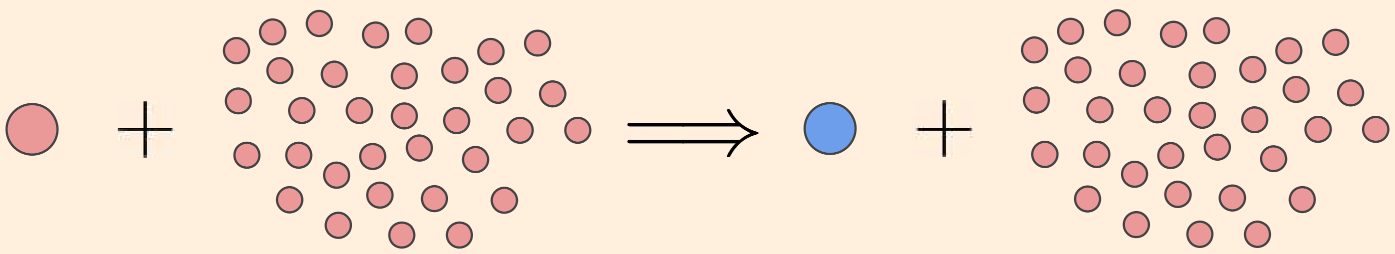
$$\tilde{\rho} = \bigotimes_{n=1}^{\tilde{N}} |\tilde{E}_{k_n}^{(n)}\rangle\langle\tilde{E}_{k_n}^{(n)}|$$

$$N \to \infty$$

$$\epsilon \simeq 1 - \Phi\left(\frac{W^{\text{diss}}}{\sigma}\right)$$

- **Beyond** incoherent states
- Fluctuation-dissipation relation
- Free energy fluctuations are just **energy** fluctuations

Why fluctuation-dissipation relations?



$$H(\lambda) = H_0 - \lambda H_0$$

$$W^{\mathrm{diss}} \simeq \beta\lambda[\langle H^2\rangle_0 - \langle H\rangle_0^2]$$

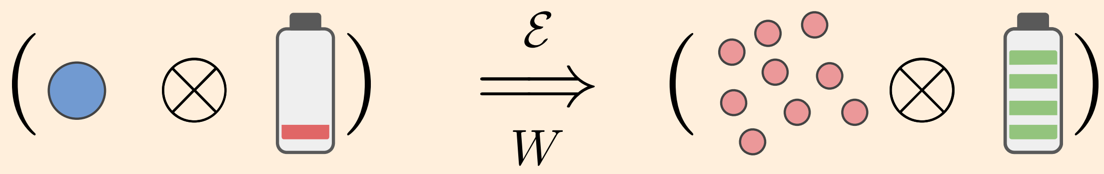
Theorem 1 and 2. Fluctuation-dissipation relation for thermodynamic distillation process:

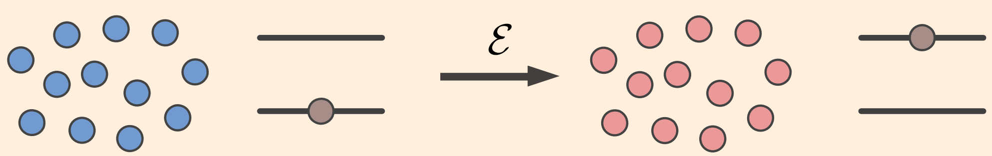
$$D(\rho\|\gamma) - D(\tilde{\rho}\|\tilde{\gamma}) \simeq \sqrt{V(\rho\|\gamma)}\,\Phi(\epsilon)^{-1}$$

Applications

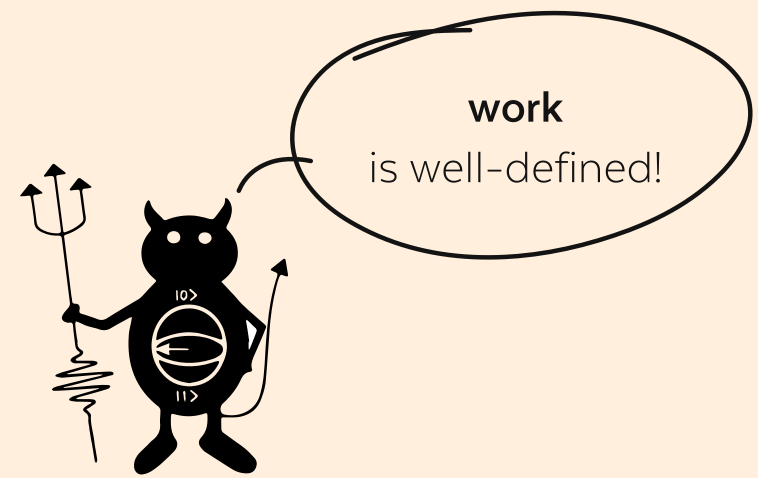
Bounds on optimal work extraction

Application of the interconversion problem

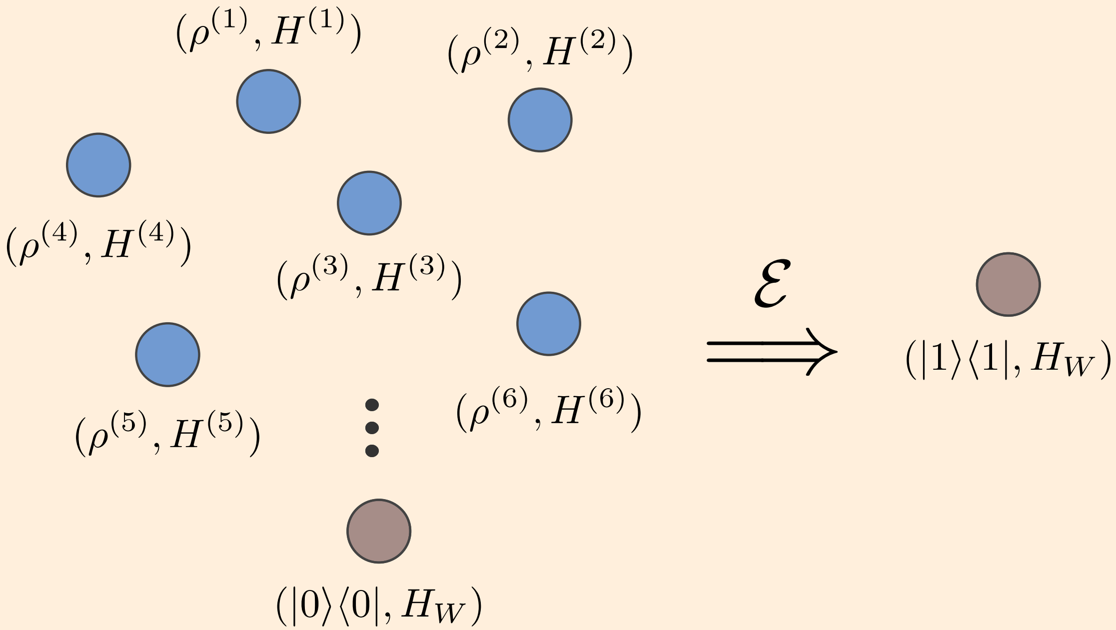


Ex. 

, $\mathcal{E}(\rho_S \otimes |0\rangle\langle 0|_B) = |W\rangle\langle W|_B$



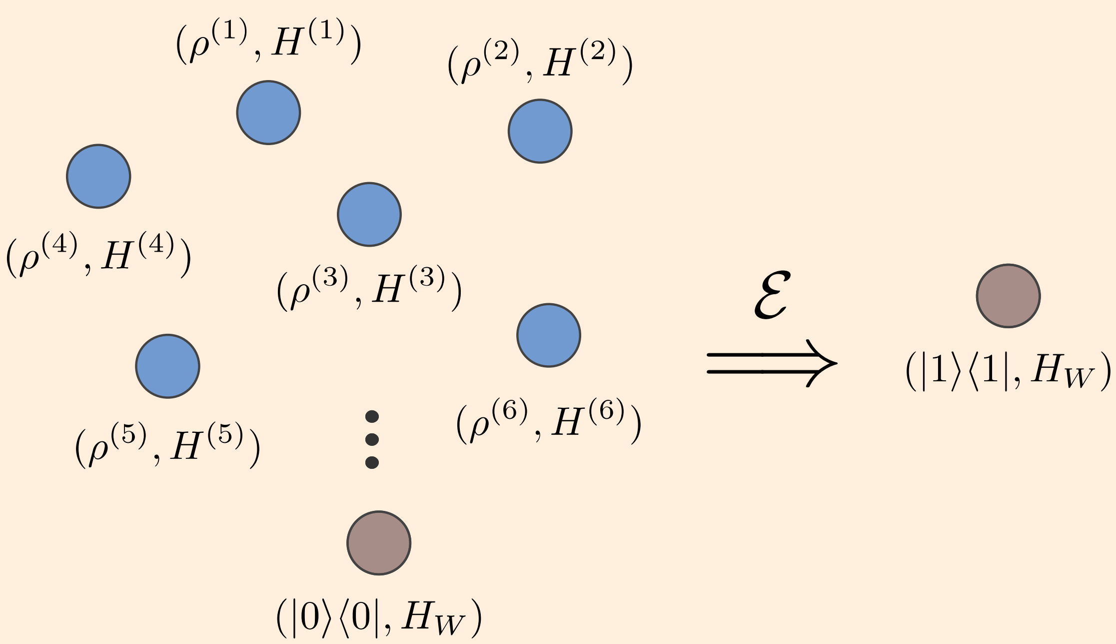
Bounds on optimal work extraction



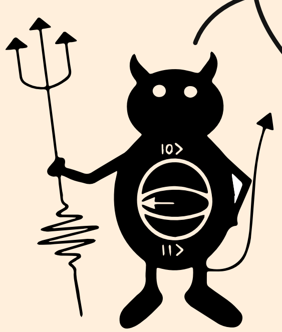
$$W^{\text{diss}} \leq \sigma\, \Phi^{-1}\left(\epsilon - \frac{C\, W(\rho\|\gamma)}{\sigma^3}\right)$$

$$H_W = 0|0\rangle\langle 0| + W|1\rangle\langle 1|_B$$

Bounds on optimal work extraction



$$\epsilon \leq 1 - \Phi\left(\frac{W_{\text{diss}}}{\sigma}\right) + \frac{C}{\sqrt{N\sigma^3}} \left| \sum_{n=1}^{N} W^*(\rho^{(n)} \| \gamma^{(n)}) \right|$$

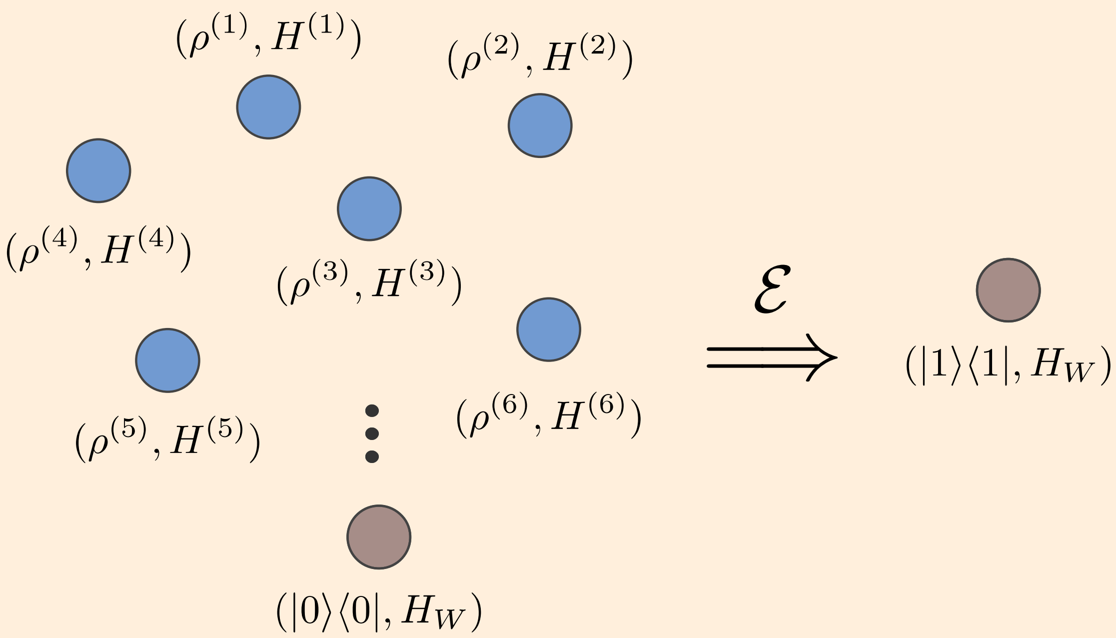


To achive the same **transformation error** from...

... states with higher σ needs to dissipate more work

...states with small σ allow one to dissipate small amounts of work

Bounds on optimal work extraction

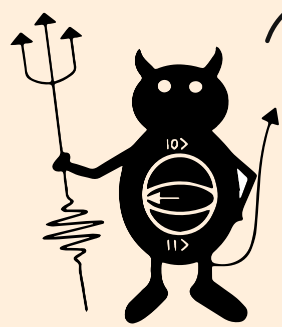


$$\epsilon \leq 1 - \Phi\left(\frac{W_{\mathrm{diss}}}{\sigma}\right) + \frac{C}{\sqrt{N\sigma^3}}\left|\sum_{n=1}^{N} W^*(\rho^{(n)}\|\gamma^{(n)})\right|$$

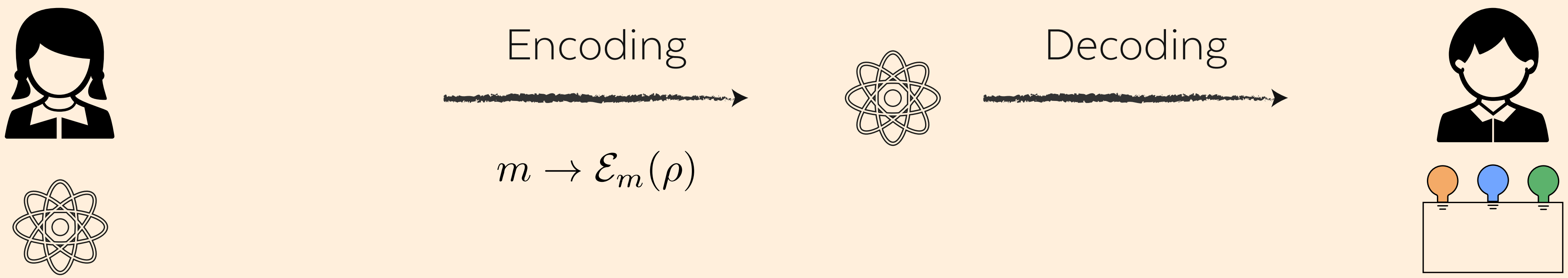
- Dissipated work in form of fluctuations
- It holds for all N
- Battery is a single system

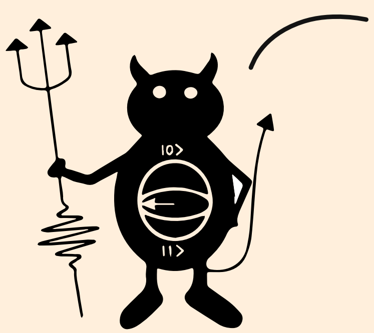
Optimal thermodynamically-free encoding of information

Given $\rho^{\otimes N}$ and being restricted to $\mathcal{E}$
how many ϵ-orthogonal states we can encode?



Optimal thermodynamically-free encoding of information



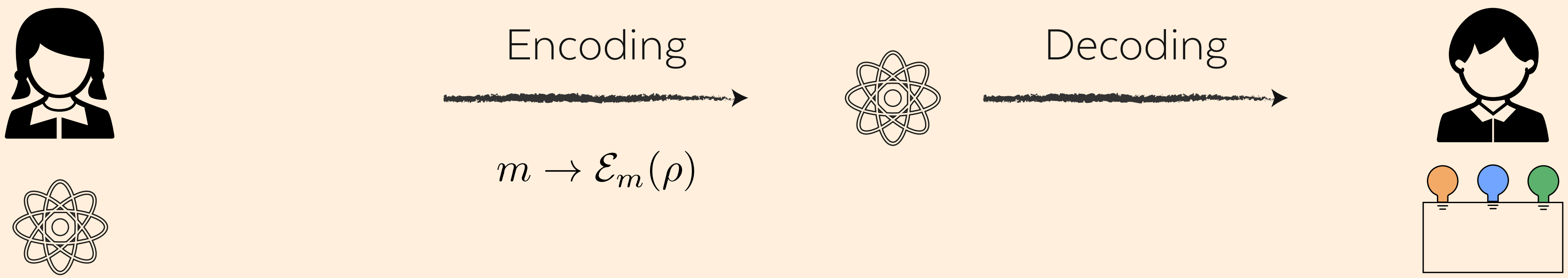


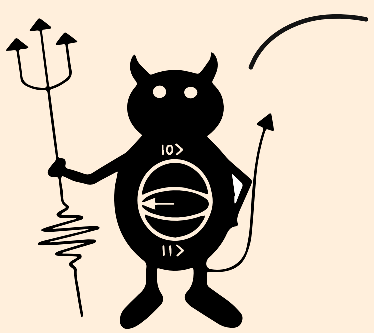
The optimal number of messages that can be encoded into ρ in a thermodynamically-free way

$$R(\sigma, N, \epsilon) := \frac{\log[M(\sigma^{\otimes N}, \epsilon)]}{N}$$

Phys. Rev. Lett. 122, 060601

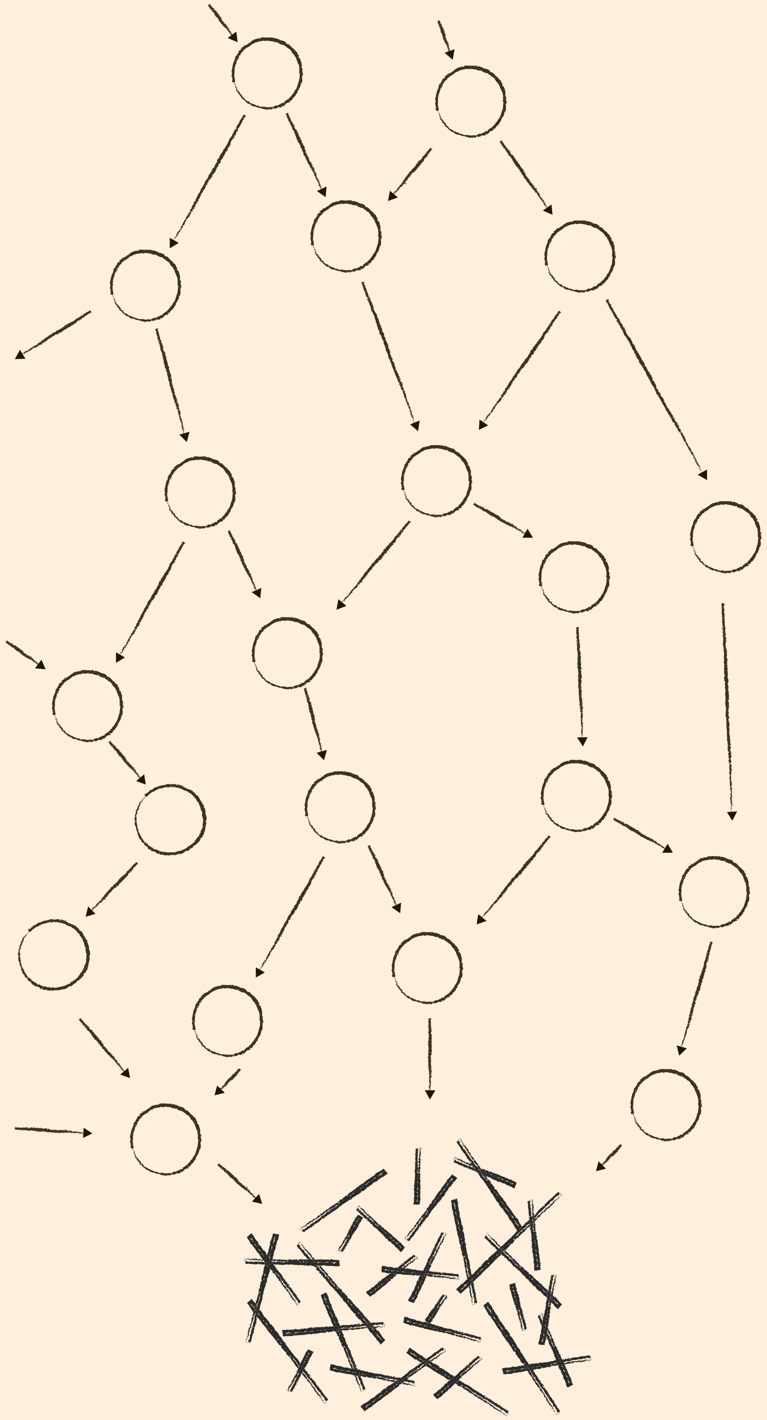
Optimal thermodynamically-free encoding of information





The optimal number of messages that can be encoded into ρ in a thermodynamically-free way

$$R(\rho, N, \epsilon) = D(\rho\|\gamma) + \frac{1}{\sqrt{N}}\sqrt{V(\rho\|\gamma)}\,\Phi^{-1}(\epsilon)$$



Outlook

I. State interconversion problem: **incoherent** and **coherent** initial states

2. **Work extraction** and **thermal ecoding** of information

3. Second-order asymptotic analysis for state transformation from **general mixed** states.

arXiv.????.????


thanks!
|0>
|1>