When to/not to quantumly simulate a classical transition?

Fereshte Shahbeigi (Jagiellonian University, Cracow, Poland)

In collaboration with:

Christopher T. Chubb (ETH, Zurich)

Ryszard Kukulski (PAS, Gliwice)

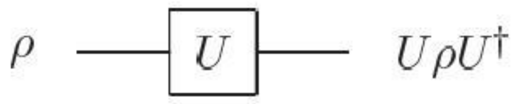
Lukasz Pawela (PAS, Gliwice)

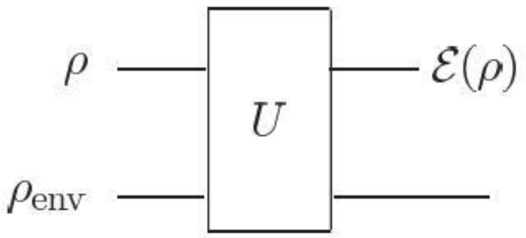
Kamil Korzekwa (JU, Cracow)

Based on arXiv:2305.17163

SMP54, NCU, Torun, 2023

Qauntum Channels:





	$\mathcal{E}(\rho) = \text{Tr}_{\text{env}}\left[U\left(\rho \otimes \sigma\right)U^\dagger\right]$	$D_{\mathcal{E}} = d(\mathcal{E} \otimes \mathcal{I})\|\psi_+\rangle\langle\psi_+\|$
TP:	$\text{Tr}[\mathcal{E}(X)] = \text{Tr}[X]$ $\forall X$	$\text{Tr}_A(D_{\mathcal{E}}) = \mathbb{I}$
HP:	$\mathcal{E}(X)^\dagger = \mathcal{E}(X^\dagger)$ $\forall X$	$D_{\mathcal{E}} = D_{\mathcal{E}}^\dagger$
CP:	$(\mathcal{E} \otimes \mathcal{I}_n)(X) \geq 0$ $\forall n \;\&\; \forall X \geq 0$	$D_{\mathcal{E}} \geq 0$

On the other hand:

An open quantum system can obey a differential equation of motion, like GKLS:

$$\frac{\mathrm{d}\rho}{\mathrm{dt}} = i[\rho, H] + \sum G_{\alpha\beta} F_\alpha \rho F_\beta^\dagger - \frac{1}{2}\{F_\beta^\dagger F_\alpha, \rho\}$$

$$\boxed{\frac{\mathrm{d}\rho}{\mathrm{dt}} = \mathcal{L}(\rho) \Longrightarrow \rho(t) = \mathrm{e}^{\mathcal{L}t}(\rho)}$$

Lindblad Operators:

$\mathcal{L}$	$D_{\mathcal{L}} = d(\mathcal{L} \otimes \mathcal{I})\|\psi_+\rangle\langle\psi_+\|$
TS: $\mathrm{Tr}[\mathcal{L}(X)] = 0 \quad \forall X$	$\mathrm{Tr}_A(D_{\mathcal{L}}) = 0$
HP: $\mathcal{L}(X)^\dagger = \mathcal{L}(X^\dagger) \quad \forall X$	$D_{\mathcal{L}} = D_{\mathcal{L}}^\dagger$
<u>Conditionally</u> Completely Positive	$\Pi D_{\mathcal{L}} \Pi \geq 0$ $\Pi = I - \|\psi_+\rangle\langle\psi_+\|$

M. M. Wolf, J. Eisert, T.S. Cubitt, and J.I. Cirac, PRL (2008)

Markovianity Problem:

A channel is Markovian if it is in the closure of the maps of the form $\mathcal{E}_t = e^{\mathcal{L}t}$.

1. M. M. Wolf, J. Eisert, T.S. Cubitt, and J.I. Cirac, PRL (2008)
2. T. S. Cubitt, J. Eisert, and M. M. Wolf, Commun. Math. Phys. (2012)
3. M. M. Wolf, and J. I. Cirac, Commun. Math. Phys. (2012)

Markovianity Problem:

A channel is Markovian if it is in the closure of the maps of the form $\mathcal{E}_t = \mathrm{e}^{\mathcal{L}t}$.

1. M. M. Wolf, J. Eisert, T.S. Cubitt, and J.I. Cirac, PRL (2008)
2. T. S. Cubitt, J. Eisert, and M. M. Wolf, Commun. Math. Phys. (2012)
3. M. M. Wolf, and J. I. Cirac, Commun. Math. Phys. (2012)

When $\mathrm{Log}\mathcal{E}$ is a valid Lindbladian, i.e., HP, TS, CCP?

Markovianity Problem:

A channel is Markovian if it is in the closure of the maps of the form $\mathcal{E}_t = e^{\mathcal{L}t}$.

1. M. M. Wolf, J. Eisert, T.S. Cubitt, and J.I. Cirac, PRL (2008)
2. **T. S. Cubitt, J. Eisert, and M. M. Wolf, Commun. Math. Phys. (2012)**
3. M. M. Wolf, and J. I. Cirac, Commun. Math. Phys. (2012)

When $\text{Log}\mathcal{E}$ is a valid Lindbladian, i.e., HP, TS, CCP?

Markovianity Problem:

A channel is Markovian if it is in the closure of the maps of the form $\mathcal{E}_t = e^{\mathcal{L}t}$.

1. M. M. Wolf, J. Eisert, T.S. Cubitt, and J.I. Cirac, PRL (2008)
2. **T. S. Cubitt, J. Eisert, and M. M. Wolf, Commun. Math. Phys. (2012)**
3. M. M. Wolf, and J. I. Cirac, Commun. Math. Phys. (2012)

When Log$\mathcal{E}$ is a valid Lindbladian, i.e., HP, TS, CCP?

- **Logarithm of a matrix:** **X** is a generator of **A** if **exp(X)=A**

$$\mathbb{I}_2 \cos\theta + i(\hat{n}.\vec{\sigma}) \sin\theta = e^{i\theta(\hat{n}.\vec{\sigma})}$$

$$\theta = 2m\pi \quad \Longrightarrow \quad \mathbb{I}_2 = e^{i2m\pi(\hat{n}.\vec{\sigma})} \qquad X = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Classical Systems:

- A general linear classical process that evolves classical systems is a stochastic process:

$$T = \left(t_{ij}\right)_{d\times d} \qquad \text{Such that} \qquad t_{ij} \geq 0,\ \sum_i t_{ij} = 1$$

Classical Systems:

- A general linear classical process that evolves classical systems is a stochastic process:

$$T = \left(t_{ij}\right)_{d\times d} \qquad \text{Such that} \qquad t_{ij} \geq 0,\ \sum_i t_{ij} = 1$$

- Memoryless equation of Motion: $\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = L\vec{p}$ where

$$\forall i \neq j :\ L_{ij} \geq 0 \quad \text{and} \quad \sum_i L_{ij} = 0$$

Classical Markovianity (Classical Embeddability):

For which T there is an L such that $T = \mathrm{e}^{L}$?

G. Elfving, Acta Soc. Sci. Fennicae, n. Ser. A2 **8**, (1937)

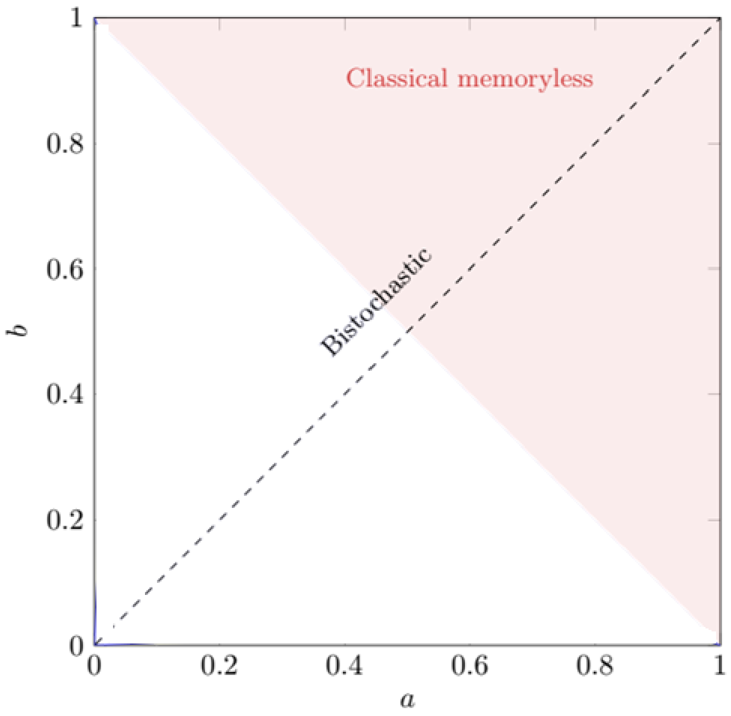
The same difficulty/complexity due to non-uniqueness of Logarithm appears

Classical Markovianity (Classical Embeddability):

For which T there is an L such that $T = e^L$?

G. Elfving, Acta Soc. Sci. Fennicae, n. Ser. A2 **8**, (1937)

The same difficulty/complexity due to non-uniqueness of Logarithm appears



$$T = \begin{pmatrix} a & 1-b \\ 1-a & b \end{pmatrix}$$

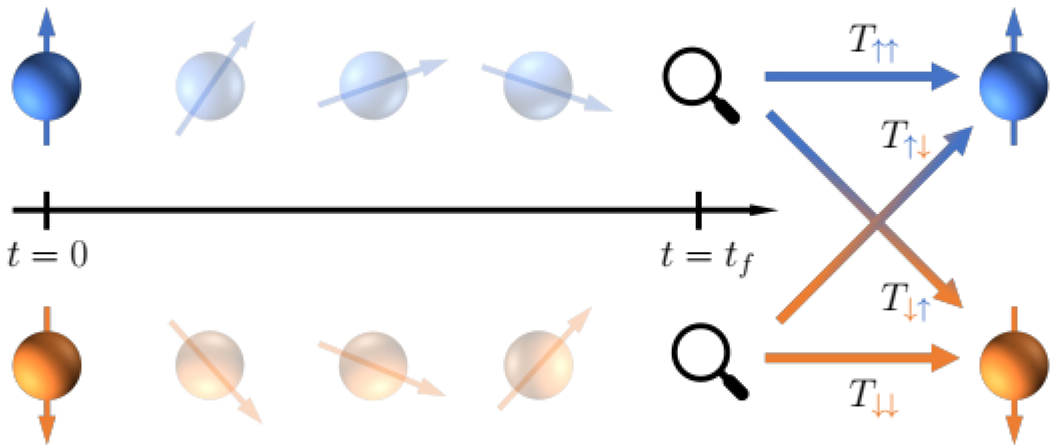
Even $T = \sigma_x$ is not Markovian

How to simulate quantumly a classical stochastic matrix

Every classical stochastic matrix is a classical action of a quantum channel

$$T_{ij} = \langle i|\mathcal{E}(|j\rangle\langle j|)|i\rangle$$

This map is surjective but not bijective. Typically, it is highly non-unique.



Quantum Embeddability:

- **Quantum Embeddability Problem:** K. Korzekwa and M. Lostaglio, PRX (2021)

T is quantum embeddable if it is the classical action of some Markovian $\mathcal{E}$.

$$T_{ij} = \langle i|\mathcal{E}(|j\rangle\langle j|)|i\rangle \quad \text{where } \mathcal{E} \text{ is Markovian}$$

Quantum Embeddability:

- **Quantum Embeddability Problem:** K. Korzekwa and M. Lostaglio, PRX (2021)

T is quantum embeddable if it is the classical action of some Markovian $\mathcal{E}$.

$$T_{ij} = \langle i|\mathcal{E}(|j\rangle\langle j|)|i\rangle \quad \text{where } \mathcal{E} \text{ is Markovian}$$

- **Why is it important?** Mamory advantage in simulation

Quantum Embeddability:

- **Quantum Embeddability Problem:** K. Korzekwa and M. Lostaglio, PRX (2021)

T is quantum embeddable if it is the classical action of some Markovian $\mathcal{E}$.

$$T_{ij} = \langle i|\mathcal{E}(|j\rangle\langle j|)|i\rangle \quad \text{where } \mathcal{E} \text{ is Markovian}$$

- **Why is it important?** Mamory advantage in simulation

- **Why is it difficult?** $T \Longrightarrow \mathcal{E} \Longrightarrow \mathcal{L}$

Quantum Embeddability:

- **Quantum Embeddability Problem:** K. Korzekwa and M. Lostaglio, PRX (2021)

T is quantum embeddable if it is the classical action of some Markovian $\mathcal{E}$.

$$T_{ij} = \langle i|\mathcal{E}(|j\rangle\langle j|)|i\rangle \qquad \text{where } \mathcal{E} \text{ is Markovian}$$

- **Why is it important?** Mamory advantage in simulation

- **Why is it difficult?** $T \Longrightarrow \mathcal{E} \Longrightarrow \mathcal{L}$

So what to do?

Results:

1. Every classical embeddable stochastic matrix is quantum embeddable.

$$T = \mathrm{e}^{Lt} \Longrightarrow \mathcal{E} = \mathrm{e}^{\mathcal{L}t}$$

K. Korzekwa and M. Lostaglio, PRX (2021)

Results:

1. Every classical embeddable stochastic matrix is quantum embeddable.

$$T = \mathrm{e}^{Lt} \Longrightarrow \mathcal{E} = \mathrm{e}^{\mathcal{L}t}$$

2. All unistochastic matrices are quantum embeddable.

$$T_{ij} = |U_{ij}|^2 \Longrightarrow \mathcal{E}(\cdot) = U(\cdot)U^\dagger$$

K. Korzekwa and M. Lostaglio, PRX (2021)

Results:

1. Every classical embeddable stochastic matrix is quantum embeddable.

$$T = \mathrm{e}^{Lt} \Longrightarrow \mathcal{E} = \mathrm{e}^{\mathcal{L}t}$$

2. All unistochastic matrices are quantum embeddable.

$$T_{ij} = |U_{ij}|^2 \Longrightarrow \mathcal{E}(\cdot) = U(\cdot)U^\dagger$$

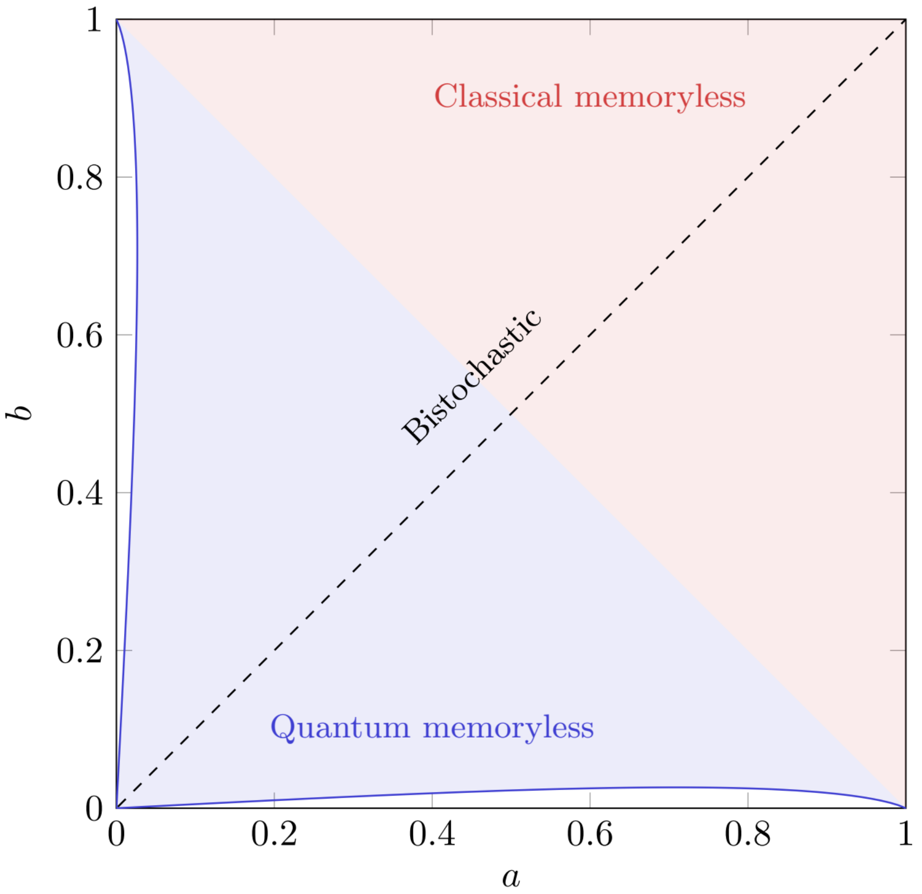
The set of quantum embeddable maps is strictly larger than the classical one

K. Korzekwa and M. Lostaglio, PRX (2021)

Results:

. Two dimensional maps: ArXiv:2305.17163

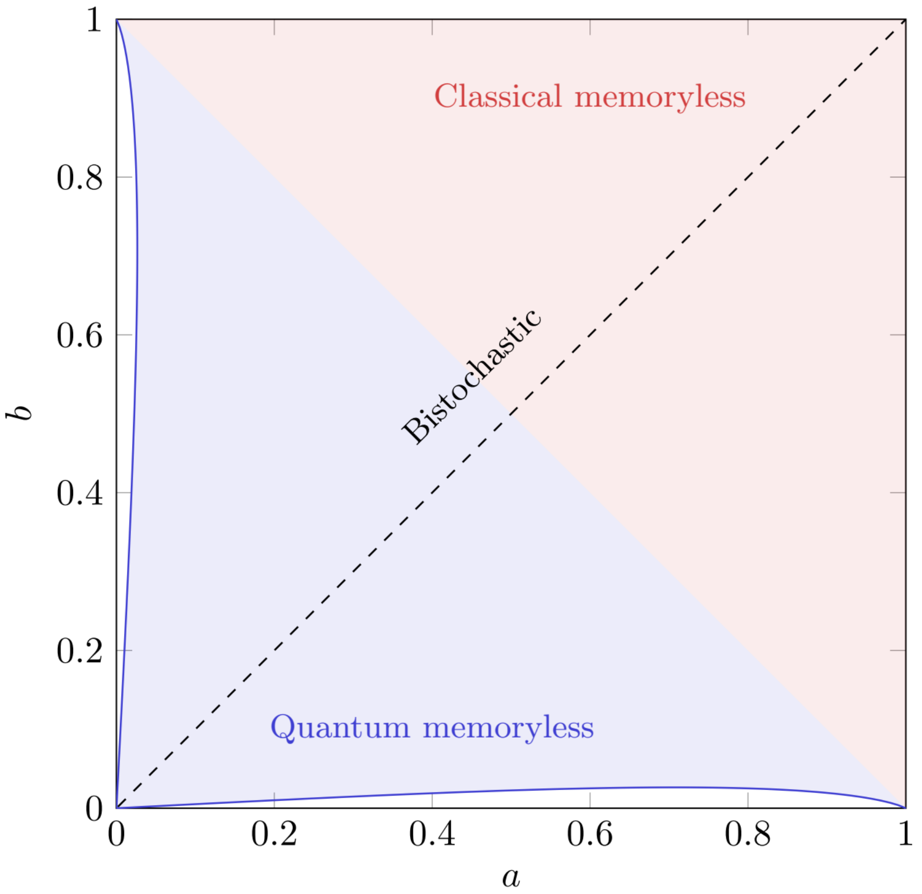
$$T = \begin{pmatrix} a & 1-b \\ 1-a & b \end{pmatrix}$$



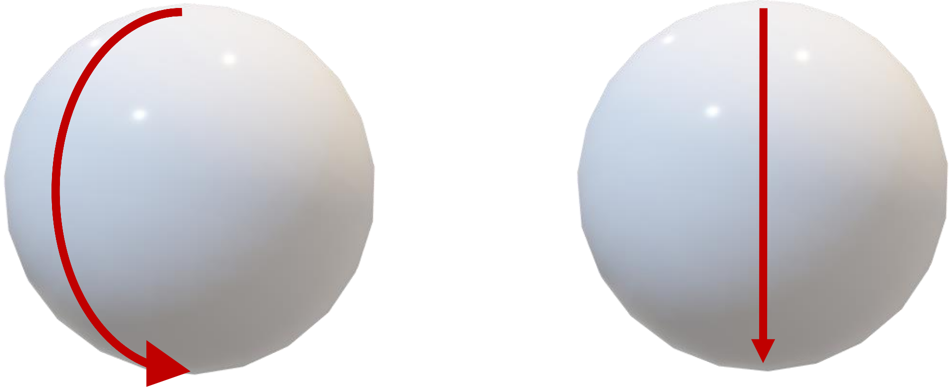
Results:

. Two dimensional maps: ArXiv:2305.17163

$$T = \begin{pmatrix} a & 1-b \\ 1-a & b \end{pmatrix}$$



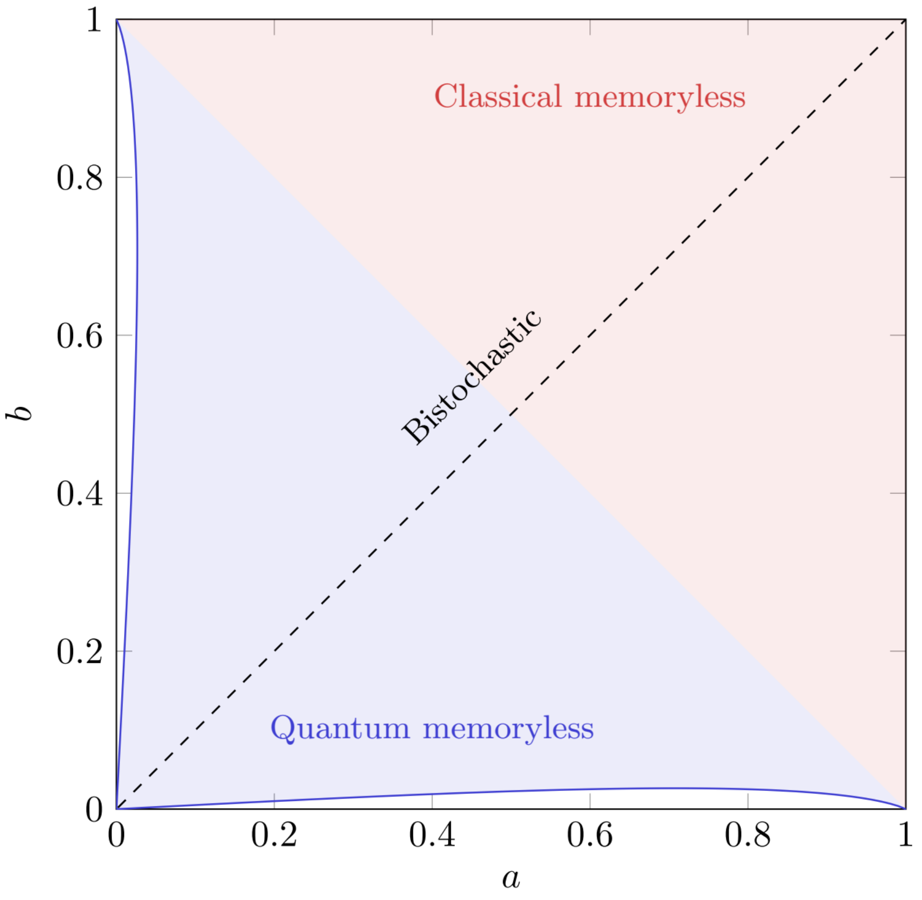
. Sketch of the Proof:



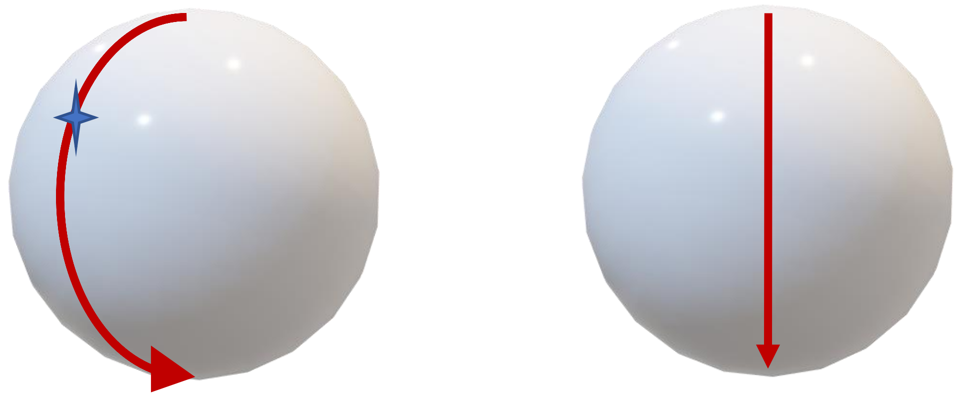
Results:

. Two dimensional maps: ArXiv:2305.17163

$$T = \begin{pmatrix} a & 1-b \\ 1-a & b \end{pmatrix}$$



. Sketch of the Proof:



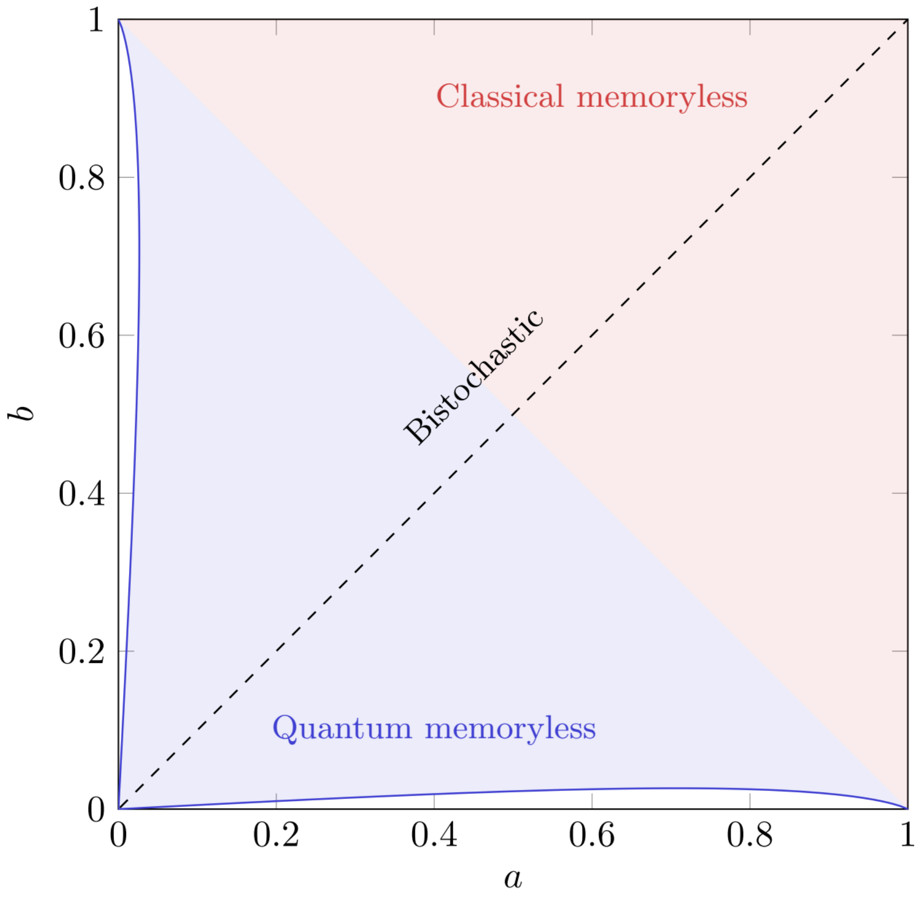
$$e^{\mathcal{L}\frac{t}{3}}(|0\rangle\langle 0|) = |\psi\rangle\langle\psi|$$

Results:

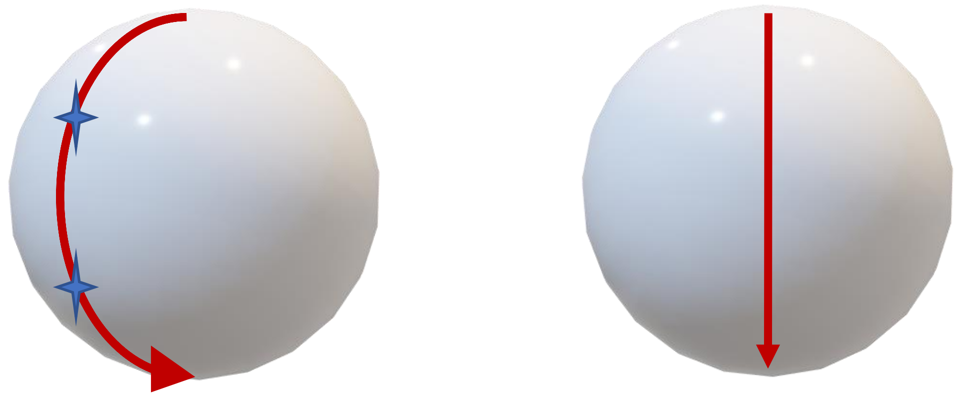
. Two dimensional maps:

ArXiv:2305.17163

$$T = \begin{pmatrix} a & 1-b \\ 1-a & b \end{pmatrix}$$



. Sketch of the Proof:



$$e^{\mathcal{L}\frac{t}{3}}(|0\rangle\langle 0|) = |\psi\rangle\langle\psi|$$

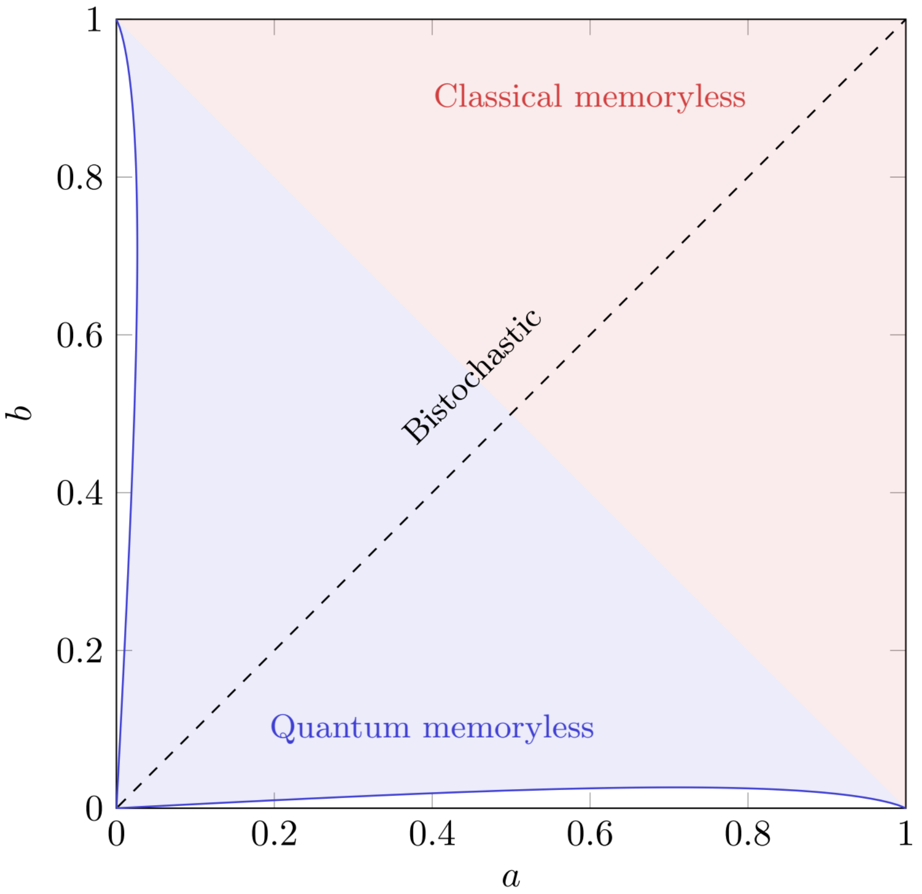
$$e^{\mathcal{L}\frac{t}{3}}(|\psi\rangle\langle\psi|) = |\phi\rangle\langle\phi|$$

Results:

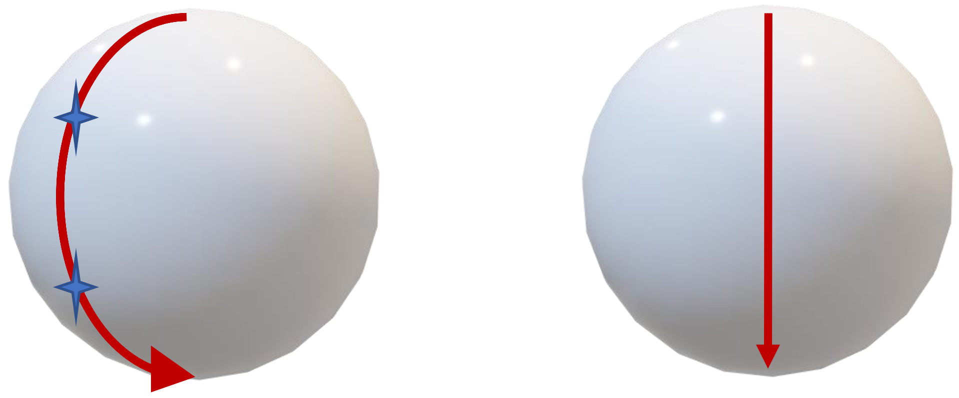
. Two dimensional maps:

ArXiv:2305.17163

$$T = \begin{pmatrix} a & 1-b \\ 1-a & b \end{pmatrix}$$



. Sketch of the Proof:



$$e^{\mathcal{L}\frac{t}{3}}(|0\rangle\langle 0|) = |\psi\rangle\langle\psi|$$

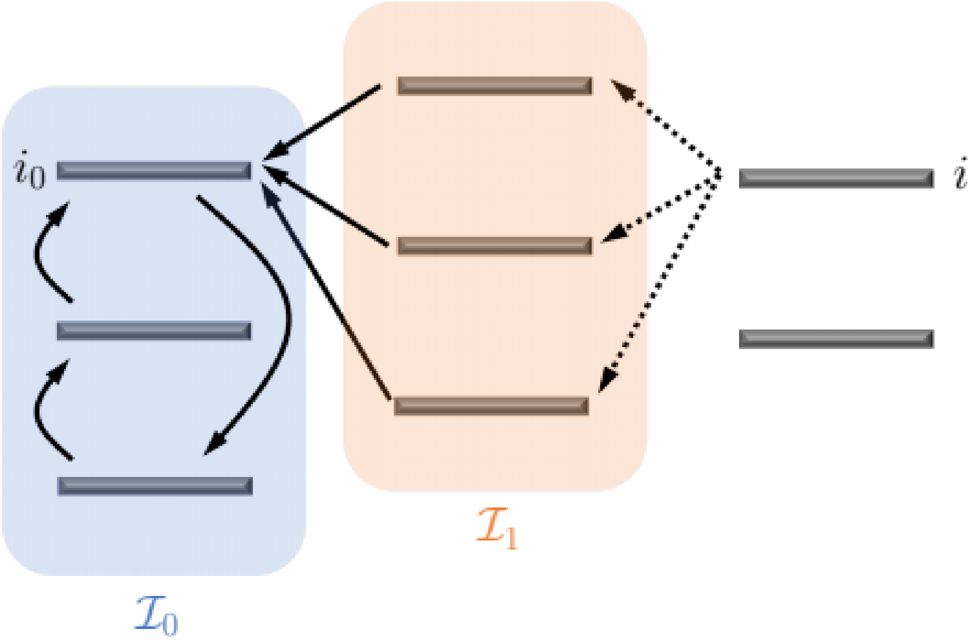
$$e^{\mathcal{L}\frac{t}{3}}(|\psi\rangle\langle\psi|) = |\phi\rangle\langle\phi|$$

$$e^{\mathcal{L}\frac{t}{3}}(|\phi\rangle\langle\phi|) = |1\rangle\langle 1|$$

Results:

. Higher dimensional maps (non-embeddable ones): ArXiv:2305.17163

Consider a $d \times d$ stochastic matrix T. Let $I_0 \subset \{1, \ldots, d\}$ be a subset of indices such that T invariantly permutes I_0. Also, let I_1 denote a subset of the complementary set of I_0, where for any $i_1 \in I_1$ it holds that $T_{i_0,i_1} = 1$ for some fixed $i_0 \in I_0$. Then, T is not quantum embeddable if there exists an index i such that $\sum_{i_1 \in I_1} T_{i_1,i} = 1$.



Results:

. Higher dimensional maps (embeddable ones): ArXiv:2305.17163

3. If T has the following form $R_1 \oplus R_2$.

Results:

. Higher dimensional maps (embeddable ones): ArXiv:2305.17163

3. If T has the following form $R_1 \oplus R_2$.

4. If T has the following form $\left(\begin{array}{c|c} R & S \\ \hline 0 & B \end{array}\right)$.

$$\mathcal{L} = \mathcal{L}_R + \gamma \mathcal{L}_S$$

Let $T_{i5} = T_{i1}$, then the above Lindbladian implies $\mathcal{E}(|1\rangle\langle 1|) \approx \mathcal{E}(|5\rangle\langle 5|)$

Results:

. Higher dimensional maps (embeddable ones): ArXiv:2305.17163

3. If T has the following form $R_1 \oplus R_2$.

4. If T has the following form $\left(\begin{array}{c|c} R & S \\ \hline 0 & B \end{array}\right)$.

$$\mathcal{L} = \mathcal{L}_R + \gamma \mathcal{L}_S$$

Let $T_{i5} = T_{i1}$, then the above Lindbladian implies $\mathcal{E}(|1\rangle\langle 1|) \approx \mathcal{E}(|5\rangle\langle 5|)$

5. An extreme stochastic matrix is quantum embeddable iff it includes a permutation as a diagonal block, and its other columns are given by copies of the columns of this permutation.

$$n(d) = \sum_{m=1}^{d} \binom{d}{m} m!\ m^{d-m}$$

Results:

. Higher dimensional maps (embeddable ones): ArXiv:2305.17163

3. If T has the following form $R_1 \oplus R_2$.

4. If T has the following form $\left(\begin{array}{c|c} R & S \\ \hline 0 & B \end{array}\right)$.

$$\mathcal{L} = \mathcal{L}_R + \gamma \mathcal{L}_S$$

Let $T_{i5} = T_{i1}$, then the above Lindbladian implies $\mathcal{E}(|1\rangle\langle 1|) \approx \mathcal{E}(|5\rangle\langle 5|)$

5. An extreme stochastic matrix is quantum embeddable iff it includes a permutation as a diagonal block, and its other columns are given by copies of the columns of this permutation.

$$n(d) = \sum_{m=1}^{d} \binom{d}{m} m!\; m^{d-m}$$

Thank you and Bon Appetit!