Quantum dichotomies and coherent thermodynamics beyond first-order asymptotics

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Background

Quantum dichotomies

- **Quantum dichotomy:** a pair of quantum states (ρ, σ).
- **Transforming quantum dichotomies:** Does there exist a quantum channel $\mathcal{E}$ such that

$$\mathcal{E}(\rho_1) = \rho_2 \quad \text{and} \quad \mathcal{E}(\sigma_1) = \sigma_2.$$

  If so, we write $(\rho_1, \sigma_1) \succeq (\rho_2, \sigma_2)$.
- **Approximate transformations:** Does there exist a quantum channel $\mathcal{E}$ such that

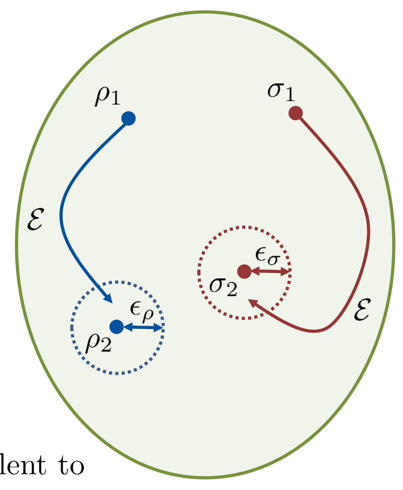
$$\delta(\mathcal{E}(\rho_1), \rho_2) \leq \epsilon_\rho \quad \text{and} \quad \delta(\mathcal{E}(\sigma_1), \sigma_2) \leq \epsilon_\sigma,$$

  where δ denotes the trace distance. If so, we write $(\rho_1, \sigma_1) \succeq_{(\epsilon_\rho, \epsilon_\sigma)} (\rho_2, \sigma_2)$.
- **Commuting dichotomies:** if $[\rho_1, \sigma_1] = [\rho_2, \sigma_2] = 0$ then $(\rho_1, \sigma_1) \succeq_{(\epsilon_\rho, \epsilon_\sigma)} (\rho_2, \sigma_2)$ is equivalent to

$$\forall x \in (\epsilon_\rho, 1): \quad \beta_x(\rho_1 \| \sigma_1) \leq \beta_{x-\epsilon_\rho}(\rho_2 \| \sigma_2) + \epsilon_\sigma,$$

  where $\beta_x(\rho \| \sigma)$ is the solution of the following semi-definite optimization problem:

$$\min_Q \mathrm{Tr}(\sigma Q), \quad \text{subject to} \quad 0 \leq Q \leq 1 \quad \text{and} \quad \mathrm{Tr}(\rho Q) \geq 1 - x.$$



Quantum thermodynamics

- **Thermal equilibrium state:** for inverse temperature β and a system with Hamiltonian H it is given by

$$\gamma = \frac{e^{-\beta H}}{\mathrm{Tr} e^{\beta H}}$$

- **Thermal operations:** quantum channels that can be written as

$$\mathcal{E}(\rho) = \mathrm{Tr}_{B'}[U(\rho \otimes \gamma_B)U^\dagger] \quad \text{with} \quad [U, H \otimes \mathbb{1}_B + \mathbb{1} \otimes H_B] = 0,$$

  where B denotes the bath system with arbitrary Hamiltonian H_B, and B' is any subsystem of the joint system.
- **Approximate transformations**: Does there exist a thermal operation $\mathcal{E}$ such that

$$\delta(\mathcal{E}(\rho_1), \rho_2) \leq \epsilon.$$

  If so, we write $\rho_1 \xrightarrow[\text{TO}]{\epsilon} \rho_2$.
- **Incoherent states**: if $[\rho_1, \gamma_1] = [\rho_2, \gamma_2] = 0$ then $\rho_1 \xrightarrow[\text{TO}]{\epsilon} \rho_2$ is equivalent to

$$\forall x \in (\epsilon, 1): \quad \beta_x(\rho_1 \| \gamma_1) \leq \beta_{x-\epsilon}(\rho_2 \| \gamma_2).$$

Problem

Asymptotic analysis in different error regimes

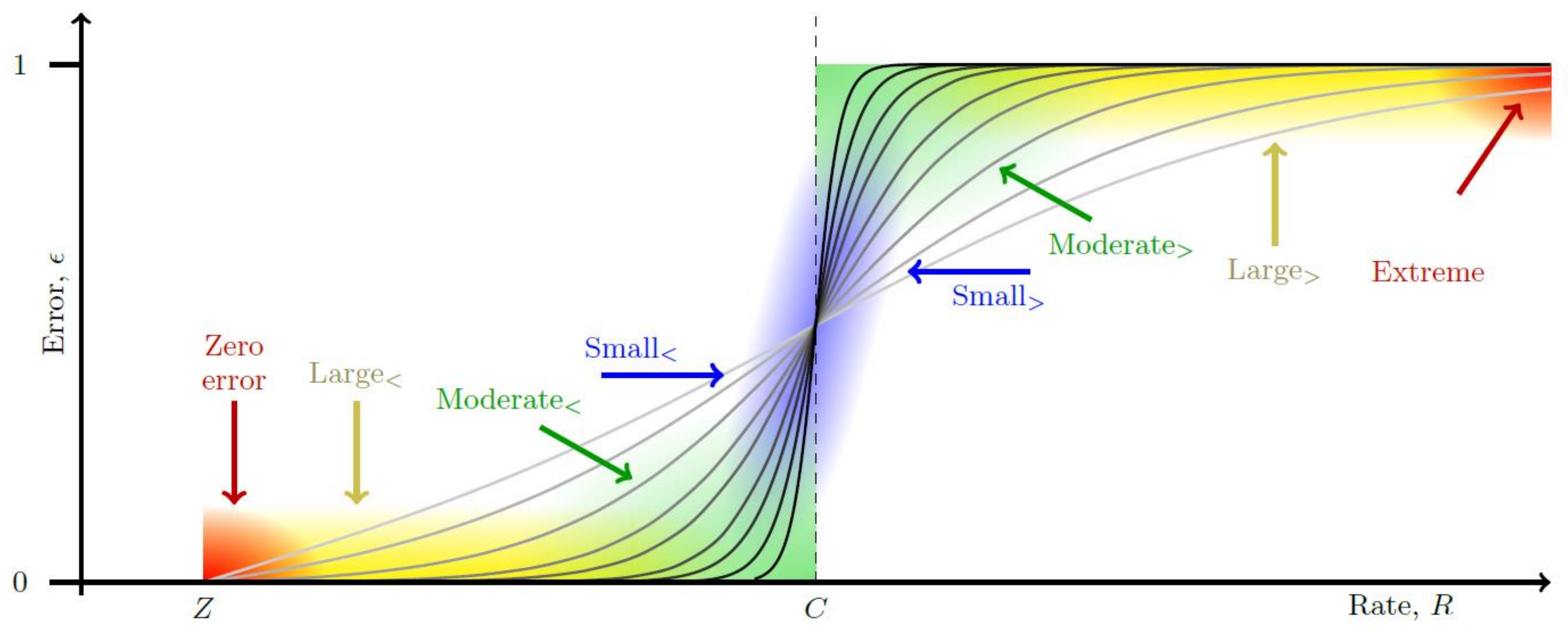
Goal 1: Find the largest possible rate $R_n^*(\epsilon_n)$ with which one can transform n copies of (ρ_1, σ_1) into copies of (ρ_2, σ_2), while allowing for at most ϵ_n in trace norm error in the first state,

$$(\rho_1^{\otimes n}, \sigma_1^{\otimes n}) \succeq_{(\epsilon_n, 0)} (\rho_2^{\otimes R_n^* n}, \sigma_2^{\otimes R_n^* n}).$$

Goal 2: Use the obtained result to find the largest possible rate $R_n^*(\epsilon_n)$ with which one can transform n copies of ρ_1 into copies of ρ_2 via thermal operations, while allowing for at most ϵ_n in trace norm error,

$$\rho_1^{\otimes n} \xrightarrow[\text{TO}]{\epsilon_n} \rho_2^{\otimes R_n^* n}.$$

Regime	**Error** (ϵ_n)	**Rate** (R_n)
Zero error	Zero	Z
Large$_<$	Approaching 0 exponentially	$[Z, C)$
Moderate$_<$	Approaching 0 subexponentially	$C - \omega(1/\sqrt{n}) \cap o(1)$
Small$_<$	Constant, < 0.5	$C - \Theta(1/\sqrt{n})$
Small$_>$	Constant, > 0.5	$C + \Theta(1/\sqrt{n})$
Moderate$_>$	Approaching 1 subexponentially	$C + \omega(1/\sqrt{n}) \cap o(1)$
Large$_>$	Approaching 1 exponentialy	(C, ∞)
Extreme	Approaching 1 superexponentially	∞



Results

Optimal rate in small deviation regime

Let $\lesssim / \simeq$ denote inequality/equality up to $o(1/\sqrt{n})$. For constant $\epsilon \in (0,1)$, and $[\rho_2, \sigma_2] = 0$, the optimal rate is

$$R_n^*(\epsilon) \simeq \frac{D(\rho_1 \| \sigma_1) + \sqrt{V(\rho_1 \| \sigma_1)/n} \cdot S_{1/\xi}^{-1}(\epsilon)}{D(\rho_2 \| \sigma_2)}.$$

Furthermore, if we consider general output dichotomies, $[\rho_2, \sigma_2] \neq 0$, then the RHS of the above till upper bounds R_n^*.

Optimal rate in moderate deviation regime

Consider an $a \in (0,1)$, and let $\lesssim / \simeq$ denote (in)equality up to terms scaling as $o\left(\sqrt{n^{a-1}}\right)$. Let $\epsilon_n := \exp(-\lambda n^a)$ for some $\lambda > 0$. For $[\rho_2, \sigma_2] = 0$ the optimal rate is

$$R_n^*(\epsilon_n) \simeq \frac{D(\rho_1 \| \sigma_1) - \left|1 - \xi^{-1/2}\right| \sqrt{2\lambda V(\rho_1 \| \sigma_1)\, n^{a-1}}}{D(\rho_2 \| \sigma_2)},$$

$$R_n^*(1 - \epsilon_n) \simeq \frac{D(\rho_1 \| \sigma_1) + \left[1 + \xi^{-1/2}\right] \sqrt{2\lambda V(\rho_1 \| \sigma_1)\, n^{a-1}}}{D(\rho_2 \| \sigma_2)}.$$

Furthermore, if we consider general output dichotomies, $[\rho_2, \sigma_2] \neq 0$, then the RHS of the above still upper bounds R_n^*.

Optimal rate in large deviation regime

For any error of the form $\epsilon_n = \exp(-\lambda n)$, for $\lambda > 0$, the optimal rate is upper bounded

$$\limsup_{n \to \infty} R_n^*(\epsilon_n) \leq \min_{-\lambda \leq \mu \leq \lambda} \overline{r}(\mu),$$

and if $[\rho_2, \sigma_2] = 0$ then it is lower bounded

$$\liminf_{n \to \infty} R_n^*(\epsilon_n) \geq \min_{-\lambda \leq \mu \leq \lambda} \widetilde{r}(\mu).$$

Optimal rate in extreme deviation regime

For $[\rho_2, \sigma_2] = 0$ the optimal zero-error rate is

$$\lim_{n \to \infty} R_n^*(0) = \min_{\alpha \in \overline{\mathbb{R}}} \frac{D_\alpha(\rho_1 \| \sigma_1)}{D_\alpha(\rho_2 \| \sigma_2)}.$$

More generally, if $[\rho_2, \sigma_2] \neq 0$, then the optimal transformation rate for all n is upper bounded by the RHS of the above.

Optimal thermodynamic protocols

- **Work extraction:** the optimal amount of work w that can be performed over a battery system W using n copies of ρ and a thermal bath at inverse temperature β,

$$\rho^{\otimes n} \otimes |0\rangle\langle 0|_W \xrightarrow[\text{TO}]{\epsilon} |1\rangle\langle 1|_W.$$

  is given by

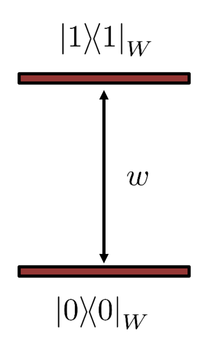
$$\frac{\beta w}{n} \simeq D(\rho \| \gamma) + \sqrt{\frac{V(\rho \| \gamma)}{n}} \Phi^{-1}(\epsilon).$$

- **Information erasure:** the minimal amount of work w needed to reset n copies of ρ,

$$\rho^{\otimes n} \otimes |0\rangle\langle 0|_W \xrightarrow[\text{TO}]{\epsilon} |0\rangle\langle 0|^{\otimes n} \otimes |1\rangle\langle 1|_W.$$

  is given by

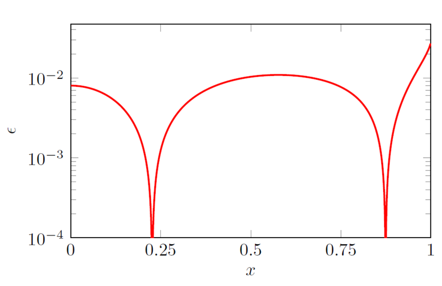
$$\frac{\beta |w|}{n} \simeq S(\rho) - \sqrt{\frac{V(\rho)}{n}} \Phi^{-1}(\epsilon).$$

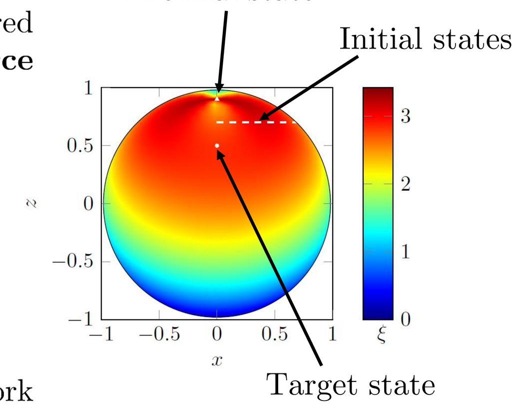


Resource resonance

When $\xi = 1$, the second-order correction to the optimal rate vanishes in the limit of zero transformation error. Thus, up to higher order terms, reversibility is restored and no dissipation takes place. We found 3 novel ways of exploiting this **resource resonance**:

- **Coherent resonance:**



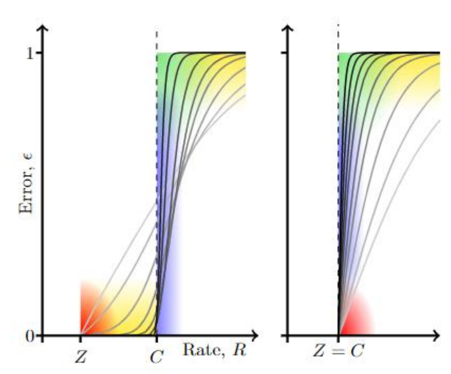


- **Work-assisted resonance:** By extracting (or investing) the amount of work w given by

$$\frac{w}{n} = \frac{1}{\beta}\left(D(\rho_1 \| \gamma_1) - \frac{V(\rho_1 \| \gamma_1)}{V(\rho_2 \| \gamma_2)} D(\rho_2 \| \gamma_2)\right),$$

  one can get the transformation rate $R = V(\rho_1 \| \gamma_1)/V(\rho_2 \| \gamma_2)$ with no dissipation, i.e., the final free energy of the system and battery will be the same as the initial one.
- **Strong resonance:** it corresponds to the situation in which the large and extreme deviation rates also collapse down to the first-order rate, which happens when:

$$\arg\min_\alpha D_\alpha(\rho_1 \| \sigma_1)/D(\rho_2 \| \sigma_2) = 1.$$



Information-theoretic and statistical notions

- Von Neumann entropy and entropy variance: $S(\rho) := -\mathrm{Tr}\rho \log \rho$, $V(\rho) := \mathrm{Tr}\rho(\log \rho)^2 - S(\rho)^2$,
- Relative entropy and relative entropy variance: $D(\rho \| \sigma) := \mathrm{Tr}\rho\,(\log \rho - \log \sigma)$, $V(\rho \| \sigma) := \mathrm{Tr}\rho\,(\log \rho - \log \sigma)^2 - D(\rho \| \sigma)^2$,
- Sandwiched relative entropy: $D_\alpha(\rho \| \sigma) := \frac{1}{\alpha - 1} \log \mathrm{Tr}\left(\sqrt{\rho}\,\sigma^{\frac{1-\alpha}{\alpha}} \sqrt{\rho}\right)^\alpha$
- Reversibility parameter: $\xi := \frac{V(\rho_1 \| \sigma_1)}{D(\rho_1 \| \sigma_1)} \Big/ \frac{V(\rho_2 \| \sigma_2)}{D(\rho_2 \| \sigma_2)}$
- Inverse of the standard normal distribution: $\Phi^{-1}(\epsilon)$
- Inverse of the sesquinormal distribution: $S_\nu^{-1}(\epsilon) = \inf_{x \in (\epsilon, 1)} \sqrt{\nu}\Phi^{-1}(x) - \Phi^{-1}(x - \epsilon)$
- For lower and upper bounds in large deviation regime, $\widetilde{r}(\mu)$ and $\overline{r}(\mu)$, refer to the arXiv paper.

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