Second-order asymptotics for transforming quantum dichotomies (with applications to quantum thermodynamics)

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Quantum dichotomies and resource theory

A quantum dichotomy is simply a pair of states (ρ, σ).

We can define the Blackwell pre-order

$$(\rho_1, \sigma_1) \succeq (\rho_2, \sigma_2) \quad \Longleftrightarrow \quad \exists\, \mathcal{E} : \ \mathcal{E}(\rho_1) = \rho_2 \text{ and } \mathcal{E}(\sigma_1) = \sigma_2$$

$$(\rho_1, \sigma_1) \succ_{(\epsilon_\rho, \epsilon_\sigma)} (\rho_2, \sigma_2) \quad \Longleftrightarrow \quad \exists\, \mathcal{E} : \ \delta\left(\mathcal{E}(\rho_1), \rho_2\right) \leq \epsilon_\rho \text{ and } \delta\left(\mathcal{E}(\sigma_1), \sigma_2\right) \leq \epsilon_\sigma$$

Specifically we look at the asymptotic trade-off between rate R_n and errors $(\epsilon_{\rho,n}, \epsilon_{\sigma,n})$ such that

$$\left(\rho_1^{\otimes n}, \sigma_1^{\otimes n}\right) \succ_{(\epsilon_{\rho,n}, \epsilon_{\sigma,n})} \left(\rho_2^{\otimes R_n n}, \sigma_2^{\otimes R_n n}\right)$$

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Several resource theories are characterised by dichotomies:

- Thermodynamics (Gibbs-preserving)

$$\sigma_1 = \sigma_2 = e^{-\beta H}/Z \qquad (\epsilon_{\rho,n}, \epsilon_{\sigma,n}) = (\epsilon_n, 0)$$

- Purity

$$\sigma_1 = \sigma_2 = I/d \qquad (\epsilon_{\rho,n}, \epsilon_{\sigma,n}) = (\epsilon_n, 0)$$

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Notation

Relative entropy: $D(\rho\|\sigma) := \operatorname{Tr}\rho\,(\log\rho - \log\sigma)$

Relative entropy variance : $V(\rho\|\sigma) := \operatorname{Tr}\rho\,(\log\rho - \log\sigma)^2 - D(\rho\|\sigma)^2$

Rényi relative entropy (Petz): $D_\alpha(\rho\|\sigma) := \frac{1}{\alpha-1}\log\operatorname{Tr}\rho^\alpha\sigma^{1-\alpha} \quad \alpha\in(0,1)$

Rényi relative entropy (Sandwiched): $\tilde{D}_\alpha(\rho\|\sigma) := \begin{cases} \frac{1}{\alpha-1}\log\operatorname{Tr}(\sqrt{\rho}\sigma^{\frac{1-\alpha}{\alpha}}\sqrt{\rho})^\alpha & \alpha>0 \\ \frac{1}{\alpha-1}\log\operatorname{Tr}(\sqrt{\sigma}\rho^{\frac{\alpha}{1-\alpha}}\sqrt{\sigma})^{1-\alpha} & \alpha<0 \end{cases}$

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Previous results

First-order asymptotics[1]:

$$\lim_{n\to\infty} R_n^*(\epsilon) = \frac{D(\rho_1\|\sigma_1)}{D(\rho_2\|\sigma_2)} =: C \qquad \forall \epsilon \in (0,1)$$

$$\lim_{n\to\infty} \epsilon_n^*(R) = \begin{cases} 0 & R < C \\ 1 & R > C \end{cases}$$

Second-order asymptotics (commuting, constant ϵ, infidelity)[2]:

$$R_n^*(\epsilon) = \frac{D(\rho_1\|\sigma_1)}{D(\rho_2\|\sigma_2)} + \sqrt{\frac{D(\rho_1\|\sigma_1)\,V(\rho_2\|\sigma_2)}{nD(\rho_2\|\sigma_2)}} Z_\nu^{-1}(\epsilon) + o(1/\sqrt{n}) \qquad \nu := \frac{V(\rho_1\|\sigma_1)\,/D(\rho_1\|\sigma_1)}{V(\rho_2\|\sigma_2)\,/D(\rho_2\|\sigma_2)}$$

Moderate deviation (commuting, sub-exponential ϵ, infidelity)[3]:

$$R_n^*\left(e^{-n^\alpha}\right) = \frac{D(\rho_1\|\sigma_1)}{D(\rho_2\|\sigma_2)} - \sqrt{\frac{2V(\rho_1\|\sigma_1)}{D(\rho_2\|\sigma_2)^2}} \left|1 - 1/\sqrt{\nu}\right| n^{(\alpha-1)/2} + o\left(n^{(\alpha-1)/2}\right) \qquad \alpha \in (0,1)$$

[1] Brandão et.al., doi:10/gf6cx3, arXiv:1111.3882
[2] Kumagai/Hayashi, doi:10/f9tvhb, arXiv:1306.4166. Chubb/Tomamichel/Korzekwa, doi:10/jdrh, arXiv:1711.01193
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Our results

- Trace distance, not infidelity
- Non-commuting inputs, partial results for non-commuting outputs

$$[\rho_1, \sigma_1] \neq 0$$

- All error regimes

$$\begin{array}{lll}
\text{Zero:} & \epsilon_n = 0 & \\
\text{Large dev. (low):} & \epsilon_n = e^{-\lambda n} & \lambda > 0 \\
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\text{Small dev.:} & \epsilon_n = \epsilon & \epsilon \in (0,1) \\
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- Two-sided error
- Simpler and more intuitive proofs

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Hypothesis testing

For a test $0 \leq P \leq I$ the type-I and type-II errors are

$$\alpha(P) := \operatorname{Tr} \rho(I - P) \qquad\qquad \beta(P) := \operatorname{Tr} \sigma P$$

Can characterise HT by the trade-off:

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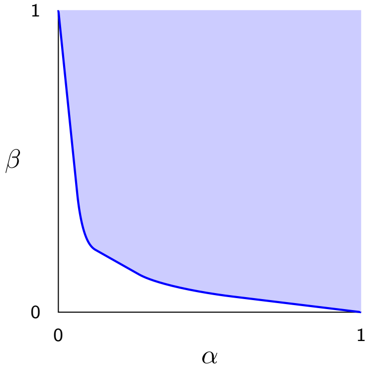

Reduction to hypothesis testing

Blackwell's equivalence theorem[4]: For commuting dichotomies

$$(\rho_1, \sigma_1) \succeq (\rho_2, \sigma_2) \qquad \Longleftrightarrow \qquad \beta_x(\rho_1 \| \sigma_1) \leq \beta_x(\rho_2 \| \sigma_2) \quad \forall x$$

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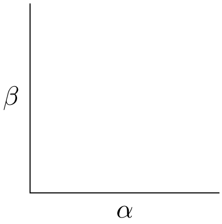

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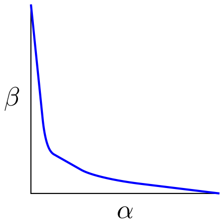

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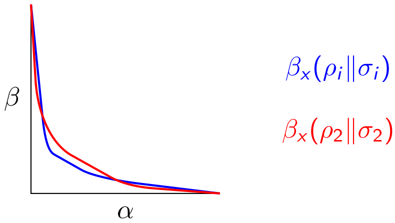

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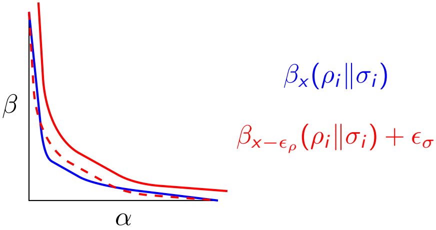

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Pinched hypothesis testing

It is known for general quantum states that

$$\beta_x(\rho_1\|\sigma_1) \leq \beta_{x-\epsilon_\rho}(\rho_2\|\sigma_2) + \epsilon_\sigma \quad \forall x \qquad \not\Longrightarrow \qquad (\rho_1,\sigma_1) \succ_{(\epsilon_\rho,\epsilon_\sigma)} (\rho_2,\sigma_2)$$

Consider the hypothesis tests between $\mathcal{P}_\sigma(\rho)$ and σ, and define

$$\tilde{\beta}_x(\rho\|\sigma) := \beta_x(\mathcal{P}_\sigma(\rho)\|\sigma)$$

If $[\rho_2,\sigma_2]=0$, then

$$\begin{aligned} &\tilde{\beta}_x(\rho_1\|\sigma_1) \leq \beta_{x-\epsilon_\rho}(\rho_2\|\sigma_2) + \epsilon_\sigma \quad \forall x \\ \Longrightarrow \qquad &(\mathcal{P}_{\sigma_1}(\rho_1),\sigma_1) \succ_{(\epsilon_\rho,\epsilon_\sigma)} (\rho_2,\sigma_2) \\ \Longrightarrow \qquad &(\rho_1,\sigma_1) \succ_{(\epsilon_\rho,\epsilon_\sigma)} (\rho_2,\sigma_2) \end{aligned}$$

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Conditions for Blackwell ordering

$$\beta_x(\rho_1\|\sigma_1) \leq \beta_{x-\epsilon_\rho}(\rho_2\|\sigma_2) + \epsilon_\sigma \quad \forall x$$

$$\Uparrow$$

$$(\rho_1, \sigma_1) \succ_{(\epsilon_\rho, \epsilon_\sigma)} (\rho_2, \sigma_2)$$

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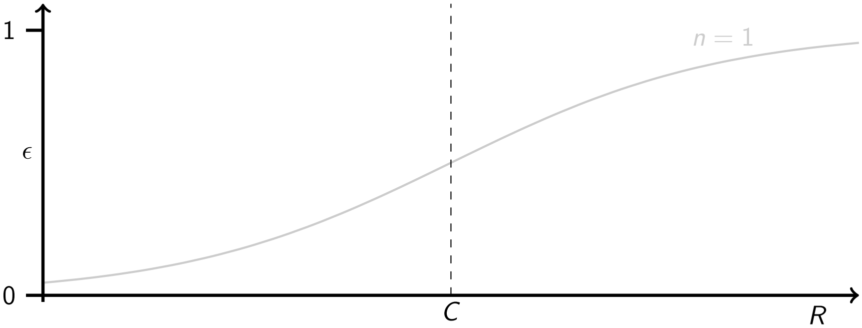
Second-order asymptotics: small, moderate, and large deviations

How fast are the convergences $R \to C$ or $\epsilon \to 0, 1$ as $n \to \infty$?



Small deviation

$R_n = C + \Theta(1/\sqrt{n})$

$\epsilon_n = \Theta(1)$

Moderate deviation

$R_n = C \mp \Theta(n^{(\alpha-1)/2})$

$\epsilon_n = e^{-\Theta(n^\alpha)}$ or $\epsilon_n = 1 - e^{-\Theta(n^\alpha)}$

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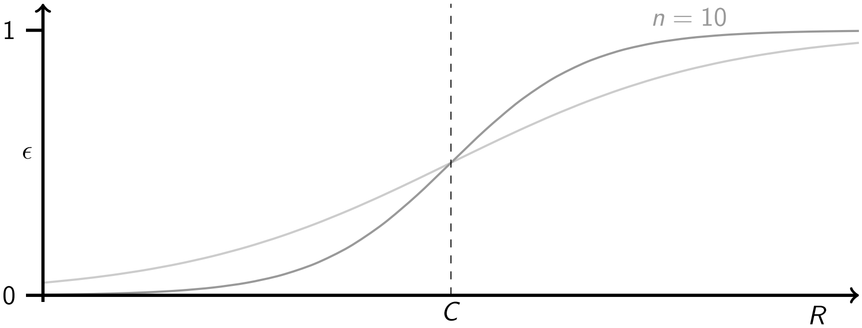

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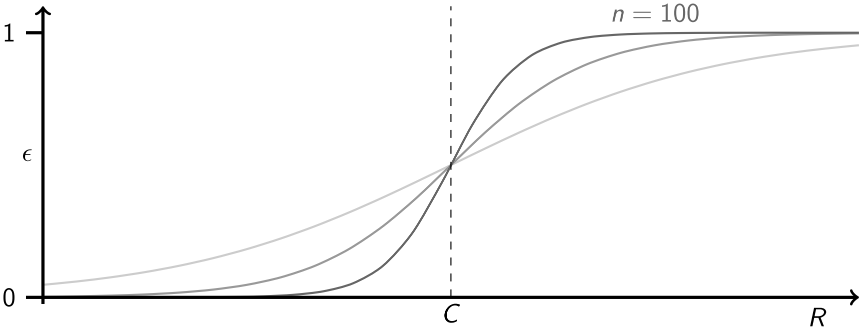

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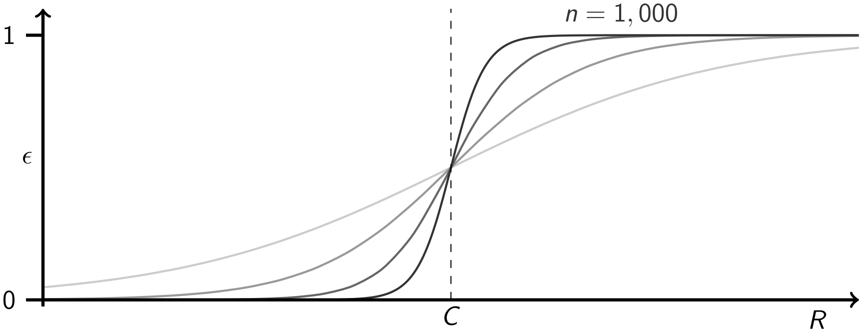

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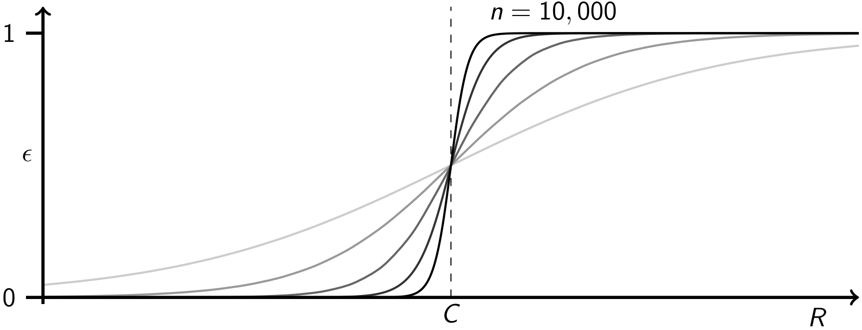

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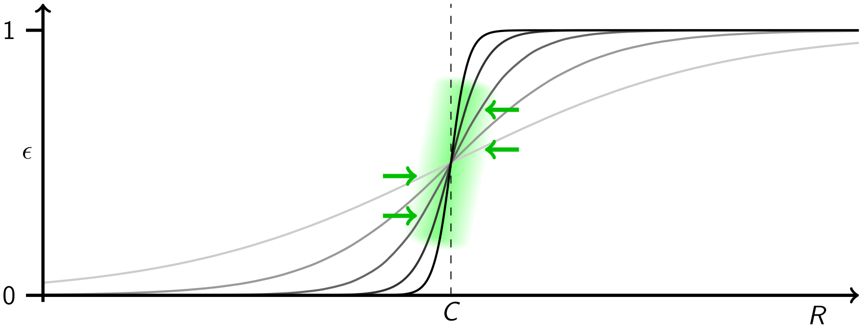

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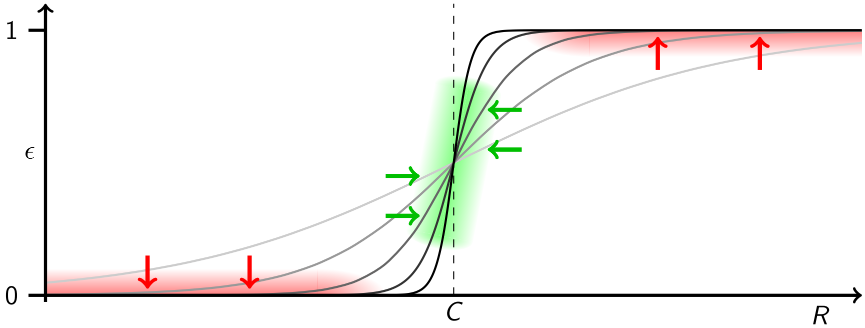

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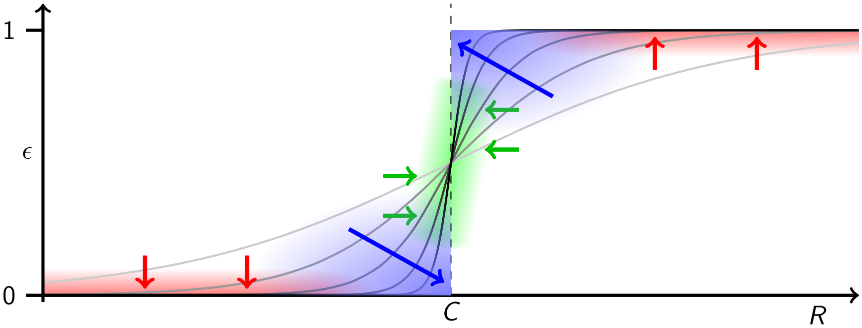

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Results (small and moderate)

Small deviation

$$R_n^*(\epsilon) = \frac{D(\rho_1\|\sigma_1)}{D(\rho_2\|\sigma_2)} + \sqrt{\frac{D(\rho_1\|\sigma_1)\, V(\rho_2\|\sigma_2)}{nD(\rho_2\|\sigma_2)}} \cdot S_\nu^{-1}(\epsilon) + o(1/\sqrt{n})$$

$$S_\nu^{-1}(\epsilon) = \inf_{x\in(\epsilon,1)} \sqrt{\nu}\Phi^{-1}(x) - \Phi^{-1}(x-\epsilon) \qquad \nu := \frac{V(\rho_1\|\sigma_1)\,/D(\rho_1\|\sigma_1)}{V(\rho_2\|\sigma_2)\,/D(\rho_2\|\sigma_2)}$$

Moderate deviation

$$R_n^*\left(e^{-n^\alpha}\right) = \frac{D(\rho_1\|\sigma_1)}{D(\rho_2\|\sigma_2)} - \sqrt{\frac{2V(\rho_1\|\sigma_1)}{D(\rho_2\|\sigma_2)^2}}\left|1-1/\sqrt{\nu}\right| n^{(\alpha-1)/2} + o\left(n^{(\alpha-1)/2}\right)$$

$$R_n^*\left(1-e^{-n^\alpha}\right) = \frac{D(\rho_1\|\sigma_1)}{D(\rho_2\|\sigma_2)} + \sqrt{\frac{2V(\rho_1\|\sigma_1)}{D(\rho_2\|\sigma_2)^2}}\left(1+1/\sqrt{\nu}\right) n^{(\alpha-1)/2} + o\left(n^{(\alpha-1)/2}\right)$$

Results (small and moderate)

Small deviation

$$R_n^*(\epsilon) = \frac{D(\rho_1\|\sigma_1)}{D(\rho_2\|\sigma_2)} + \sqrt{\frac{D(\rho_1\|\sigma_1)\,V(\rho_2\|\sigma_2)}{nD(\rho_2\|\sigma_2)}} \cdot S_\nu^{-1}(\epsilon) + o(1/\sqrt{n})$$

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Results (large)

Large deviation (high error)

$$R_n^*(1-e^{-\lambda n}) \to \inf_{t_1>1} \inf_{0<t_2<1} \frac{\widetilde{D}_{t_1}(\rho_1\|\sigma_1)+\left(\frac{t_2}{1-t_2}+\frac{t_1}{t_1-1}\right)\lambda}{D_{t_2}(\rho_2\|\sigma_2)}$$

Large deviation (low error)

$$\limsup_{n\to\infty} R_n^*(e^{-\lambda n}) \le \inf_{-\lambda<\mu<\lambda} r(\mu)$$

$$\liminf_{n\to\infty} R_n^*(e^{-\lambda n}) \ge \inf_{-\lambda<\mu<\lambda} \widetilde{r}(\mu)$$

$$r_1(\mu) := \sup_{t_2<0} \sup_{t_1<0} \frac{-\widetilde{D}_{t_1}(\rho_1\|\sigma_1)+\left(\frac{t_1}{t_1-1}-\frac{t_2}{t_2-1}\right)\mu}{-\widetilde{D}_{t_2}(\rho_2\|\sigma_2)}$$

$$r_2(\mu) := \inf_{0<t_2<1} \sup_{0<t_1<1} \frac{D_{t_1}(\rho_1\|\sigma_1)+\left(\frac{t_1}{1-t_1}-\frac{t_2}{1-t_2}\right)\mu}{D_{t_2}(\rho_2\|\sigma_2)}$$

$$r_2'(\mu) := \inf_{0<t_2<1} \sup_{0<t_1<1} \frac{\widetilde{D}_{t_1}(\rho_1\|\sigma_1)+\left(\frac{t_1}{1-t_1}-\frac{t_2}{1-t_2}\right)\mu}{D_{t_2}(\rho_2\|\sigma_2)}$$

$$r_3(\mu) := \sup_{t_2>1} \inf_{t_1>1} \frac{\widetilde{D}_{t_1}(\rho_1\|\sigma_1)+\left(\frac{t_1}{t_1-1}-\frac{t_2}{t_2-1}\right)\mu}{\widetilde{D}_{t_2}(\rho_2\|\sigma_2)}$$

$$r(\mu) := \begin{cases} r_1(\mu) & \mu < -D(\sigma_1\|\rho_1) \\ r_2(\mu) & -D(\sigma_1\|\rho_1) < \mu < 0 \\ r_3(\mu) & \mu > 0 \end{cases}$$

$$\widetilde{r}(\mu) := \begin{cases} r_1(\mu) & \mu < -D(\sigma_1\|\rho_1) \\ r_2'(\mu) & -D(\sigma_1\|\rho_1) < \mu < 0 \\ r_3(\mu) & \mu > 0 \end{cases}$$

Coherent thermodynamics

Consider the coherent transformation:

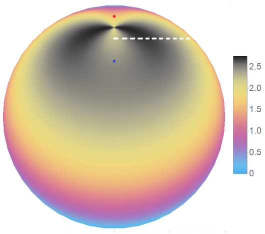
$$\rho_1 = \begin{pmatrix} 0.85 & \sqrt{0.85 \cdot 0.15x} \\ \sqrt{0.85 \cdot 0.15x} & 0.15 \end{pmatrix}$$

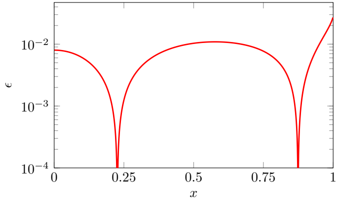
$$\rho_2 = \begin{pmatrix} 0.75 & \\ & 0.25 \end{pmatrix}$$

$$\gamma = \begin{pmatrix} 0.95 & \\ & 0.05 \end{pmatrix}$$

What is the error ϵ associated with performing this transform at rate C?

$V(\rho\|\gamma)\,/D(\rho\|\gamma)$





Summary

- Asymptotic analysis for transformation of quantum dichotomies
- Second-order analysis in all error regimes for trace distance
- Tight for general non-commuting inputs in all-but-one regime
- Opens door to study role of coherence in resource theories like thermodynamics

Thanks for listening!

`christopherchubb.com/BIID2022.pdf`