Quantumly (non-)Markovian Classical Maps

Fereshte shahbeigi

In collaboration with:

Christopher T. Chubb (ETH, Zurich)

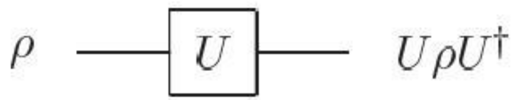
Ryszard Kukulski (PAS, Gliwice)

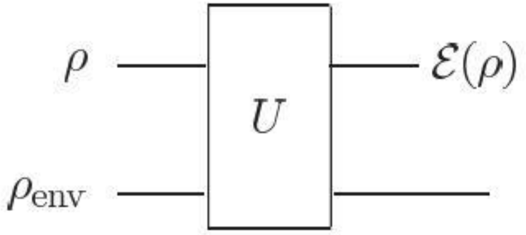
Lukasz Pawela (PAS, Gliwice)

Kamil Korzekwa (JU, Cracow)

NCU, Torun, Fall 2022

Qauntum Channels:





	$\mathcal{E}(\rho) = \text{Tr}_{\text{env}}\left[U\left(\rho \otimes \sigma\right)U^\dagger\right]$	$D_{\mathcal{E}} = d(\mathcal{E} \otimes \mathcal{I})\vert\psi_+\rangle\langle\psi_+\vert$
TP:	$\text{Tr}[\mathcal{E}(X)] = \text{Tr}[X]$ $\forall X$	$\text{Tr}_A(D_{\mathcal{E}}) = \mathbb{I}$
HP:	$\mathcal{E}(X)^\dagger = \mathcal{E}(X^\dagger)$ $\forall X$	$D_{\mathcal{E}} = D_{\mathcal{E}}^\dagger$
CP:	$(\mathcal{E} \otimes \mathcal{I}_n)(X) \geq 0$ $\forall n$ & $\forall X \geq 0$	$D_{\mathcal{E}} \geq 0$

On the other hand:

An open quantum system can obey a differential equation of motion, like GKLS:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = i[\rho, H] + \sum G_{\alpha\beta} F_\alpha \rho F_\beta^\dagger - \tfrac{1}{2}\{F_\beta^\dagger F_\alpha, \rho\}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \mathcal{L}(\rho)$$

Lindblad Operators:

	$\mathcal{L}$	$D_{\mathcal{L}} = d(\mathcal{L} \otimes \mathcal{I})\|\psi_+\rangle\langle\psi_+\|$
TS:	$\mathrm{Tr}[\mathcal{E}(X)] = 0$ $\forall X$	$\mathrm{Tr}_A(D_{\mathcal{L}}) = 0$
HP:	$\mathcal{L}(X)^\dagger = \mathcal{L}(X^\dagger)$ $\forall X$	$D_{\mathcal{L}} = D_{\mathcal{L}}^\dagger$
	<u>Conditionally</u> **Completely Positive**	$\Pi D_{\mathcal{L}} \Pi \geq 0$ $\Pi = I - \|\psi_+\rangle\langle\psi_+\|$

M. M. Wolf, J. Eisert, T.S. Cubitt, and J.I. Cirac, PRL (2008)

Markovianity Problem:

For which $\mathcal{E}$ there is an $\mathcal{L}$ and $t > 0$ such that: $\mathcal{E} = \mathrm{e}^{\mathcal{L}t}$

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2. T. S. Cubitt, J. Eisert, and M. M. Wolf, Commun. Math. Phys. (2012)
3. M. M. Wolf, and J. I. Cirac, Commun. Math. Phys. (2012)

For which $\mathcal{E}$ the entire trajectory of $\mathrm{e}^{t\mathrm{Log}\mathcal{E}}$ belongs to the set of quantum channels?

When $\mathrm{Log}\mathcal{E}$ is a valid Lindbladian, i.e., (HP, TS, CCP)?

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Complexity of the problem:

- **Logarithm of a number:** $Z = r e^{i(2m\pi+\theta)} \longrightarrow \log Z = \log r + i(2m\pi + \theta)$

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- **Logarithm of a matrix:** **X** is a generator of **A** if **exp(X)=A**

$$\mathbb{I}_2 \cos\theta + i(\hat{n}.\vec{\sigma}) \sin\theta = \mathrm{e}^{i\theta(\hat{n}.\vec{\sigma})}$$

$$\theta = 2m\pi \quad \Longrightarrow \quad \mathbb{I}_2 = \mathrm{e}^{i2m\pi(\hat{n}.\vec{\sigma})} \qquad X = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathbb{I}_3 = \mathrm{e}^X \qquad X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad X_2 = \begin{pmatrix} 0 & 2\pi & 1 \\ -2\pi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad X_3 = \begin{pmatrix} 0 & 2\pi - 1 & 1 \\ -2\pi & 0 & 0 \\ -2\pi & 0 & 0 \end{pmatrix}$$

Classical Systems:

- A general linear classical process that evolves classical systems is a stochastic process:

$$T = \left(t_{ij}\right)_{d\times d} \qquad \text{Such that} \qquad t_{ij} \geq 0,\ \sum_i t_{ij} = 1$$

- Memoryless equation of Motion: $\frac{\mathrm{d}\vec{p}}{\mathrm{dt}} = L\vec{p}$ where

$$\forall i \neq j:\ L_{ij} \geq 0 \quad \text{and} \quad \sum_i L_{ij} = 0$$

Classical Markovianity (Classical Embeddability):

For which T there is an L such that: $T = \mathrm{e}^{L}$

G. Elfving, Acta Soc. Sci. Fennicae, n. Ser. A2 **8**, (1937)

The same difficulty/complexity due to non-uniqueness of Logarithm appears

How to simulate quantumly a classical stochastic matrix

Every Classical Stochastic Matrix is Classical Action of a Quantum Channel

$$T_{ij} = \langle i|\mathcal{E}(|j\rangle\langle j|)|i\rangle$$

And Now, Quantum Embeddability:

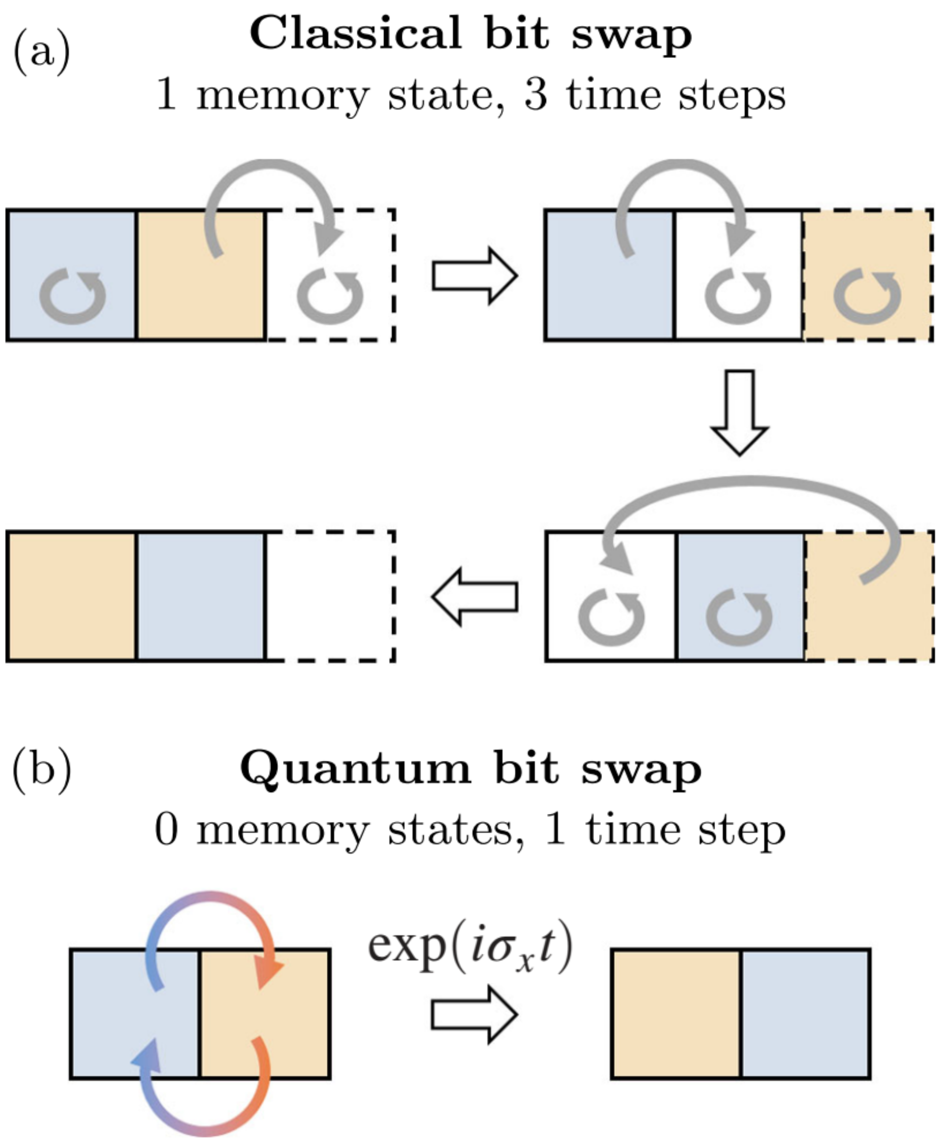
- **Quantum Embeddability Problem:** K. Korzekwa and M. Lostaglio, PRX (2021)

For which classical stochastic matrix T one can assign a quantum channel $\mathcal{E}_T$ such that

$$\frac{d}{dt}\hat{\mathcal{E}}_T(t) = \hat{\mathcal{L}}(t)\hat{\mathcal{E}}_T(t), \qquad \hat{\mathcal{E}}_T(0) = \hat{\mathcal{I}},$$

That is $\mathcal{E}_T$ is Markovian.

Simulation of a quantum and a classical bit-swap; The Motivation:



K. Korzekwa and M. Lostaglio, PRX (2021)

- **Markovian Evolutions are Memoryless**

Results:

- Every classical embeddable transition matrix is quantum embeddable

Even more:

- The set of quantum embeddable transition matrices is ***strictly*** larger than classical one, e.g., nontrivial permutations.

- All the unistochastic matrices are quantum embeddable (by a unitary evolution).

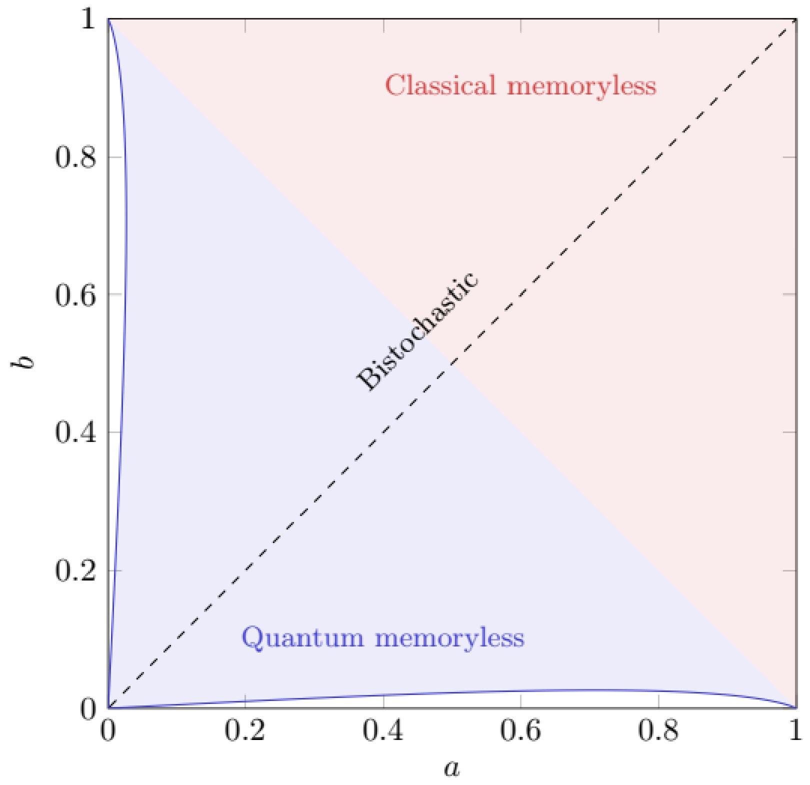
K. Korzekwa and M. Lostaglio, PRX (2021)

How to restrict the set from above?

What transitions are quantum non-embeddable?

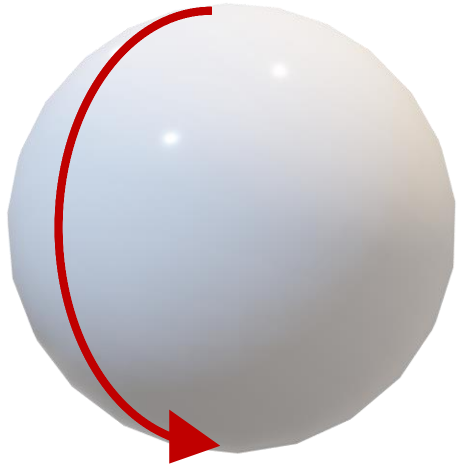
. Two dimensional maps:

$$T = \begin{pmatrix} a & 1-b \\ 1-a & b \end{pmatrix}$$

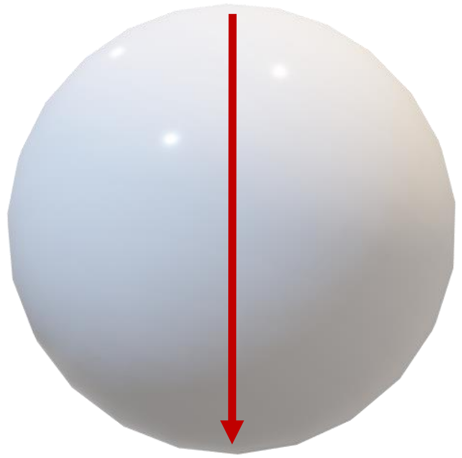


. Sketch of the Proof:

(a)



(b)



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- **Arbitrary dimension:** Consider a $d \times d$ stochastic matrix T with all columns except for column i given by the corresponding $(d-1)$ columns of a $d \times d$ permutation matrix Π with $\Pi_{ij} = 1$ for some $j \neq i$. Then, T is non-embeddable unless $T = \Pi$, or $T_{ki} = \delta_{ki}$ and $T_{jk} = 0$ for all k.

For d=i=3

$$\left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 1 & b \\ 1 & 0 & c \end{pmatrix}, \begin{pmatrix} 1 & 0 & a \\ 0 & 0 & b \\ 0 & 1 & c \end{pmatrix}, \begin{pmatrix} 0 & 1 & a \\ 0 & 0 & b \\ 1 & 0 & c \end{pmatrix}, \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{pmatrix} \right\}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Conclusion:

- More often than not, simulate your processes in quantum regime :)

Thanks for your time and patience